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# Simultaneous Adjustment of Quantities and Prices: an Example of Hamiltonian Dynamics.

(Provisional)

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Abstract

In a well-know essay, first published in 1953, ("Static and Dynamic Linear General Equilibrium Models"), Richard Goodwin analyses the dynamic adjustment of quantities and prices to long period equilibrium, in a set of n "Walrasian" markets. He treats the cross adjustment of prices and quantities as a linear Hamiltonian vector field. In more recent works Goodwin has introduced non-linear perturbations in his multisectorial adjustment model. He assumes that real consumption depends non-linearly on relative prices. This paper shows that: 1) Goodwin's behavioural hypotheses are compatible with the assumption that agents maximize; 2) if the dynamic process is Hamiltonian, symplectic coordinates changes are essential tools of analysis; 3) if real wage is rigid and returns to scale are not constant, the Hamiltonian model can generate chaotic transients or, in extreme cases, pure chaotic motions.

## 1. **Introduction**

In a pioneering work of 1953 ("Static and Dynamic Linear General Equilibrium Models") Richard Goodwin presents a leading analysis of the dynamic adjustments of quantities and prices to their long period equilibrium values. He takes two different kinds of adjustment processes into consideration: the uncoupled adjustment of prices and quantities, i.e. a gradient vector field in which prices react to excess profits per unit of output, and quantities react to excess demands per unit of output, and the crossed adjustment of prices and quantities, i.e. a Hamiltonian vector field in which the time derivative of prices depends on excess demands per unit of output, and the time derivative of gross quantities depends on excess profits per unit of output. In both cases the properties of the dynamic processes are analysed by treating the set of intersectoral demand coefficients as a linear operator on its generalized eigenspace.

In more recent works (for example "Swinging along the Autostrada: Cyclical Fluctuations along the Von Neumann Ray", (1989) or in "Chaotic Economic Dynamics", (1990)), Richard Goodwin introduces some elements of non-linearity in his basis multisectoral model; by assuming that the wage is fixed in nominal terms, he lets real consumption and investment demand depend non-linearly on relative prices. Moreover, in "The Dynamics of a Capitalist Economy", (1987), the linear crossed-dual adjustment process is represented by a Hamiltonian vector field in symplectic coordinates. This opens a wide range of fascinating potential developments for the analysis of adjustment processes in multisectoral systems subject to real perturbations.

This paper has two aims: firstly, it aims to investigate whether the behavioural hypotheses adopted by Goodwin are compatible with the assumption that agents maximize and, if so, what is the nature of their respective objective functionals. Secondly, it aims to analyse the dynamic behaviour of his 1987 Hamiltonian model, subject to the non-linear perturbation he suggests in "Chaotic

Economic Dynamics", and to point out some of the fascinating developments this approach may lead to.

In paragraph 2, Goodwin's 1953 cross-dual model is derived from an optimal control model, the objective functional of which is given by the aggregate excess profits function, i.e. by the sum of sectoral excess profits. In paragraph 3, the resulting Hamiltonian vector field is transformed, by a symplectic change of coordinates, into Goodwin's 1987 linear model. This model generates simple harmonic motions, which are illustrated by a two sector numerical example. Hence the long period equilibrium solution is stable, but the model is structurally unstable. In order to simplify the subsequent analysis of the linear model subject to non-linear perturbations, a further symplectic transformation to action angle coordinates (non-linear polar coordinates) is performed.

In paragraph 4, the following assumptions are introduced: a) the nominal wage is fixed and entirely spent for consumption; b) consumer utility functions are Cobb-Douglas; c) consumers aim to maximize their current utility. Under these hypotheses, Goodwin's 1990 consumer demand functions are easily derived. Since prices and the real wage are flexible, the resulting perturbed Hamiltonian may have equilibria which are sinks. Hence, in the numerical example, the closed orbits solution disappears and the long period equilibrium solution becomes asymptotically stable.

In the last paragraph, the analysis is restricted to a two sector model. Goodwin's basis hypotheses are slightly modified. It is assumed that the unperturbed two degree of freedom Hamiltonian is non-linear and has a homoclinic orbit. It is further assumed that the perturbation is itself periodic. By applying Melnikov's method, it is then proved that the perturbed Hamiltonian system has transverse homoclinic orbits and, therefore, Smale horseshoes.

## 2. **Goodwin's "Walrasian" model as an optimal control model.**

Basic assumptions:

a) Technology in the n-sectors economic system can be represented as a linear operator in a Euclidean n-space E. The associated matrix in standard coordinates is $A=[a_{ij}]$, $i=1...n$, $j=1...n$.

b) The economic system as a whole pursues the aim of maximizing excess profits in the time interval $t_0$-$t_1$, $t\in\mathbb{R}$, i.e.

$$\max_{q} J(t,q) = \int_{t_0}^{t_1} F(t,p,q)\,dt = \int_{t_0}^{t_1} p^t(t)(I-A)q(t)\,dt \qquad (2.1)$$

where $p^t(t)\in E^*$ and $q(t)\in E$ are the n-dimensional vectors of the deviations of relative prices and gross quantities from their long period equilibrium values and $E^*$ is the dual space of E.

c) Relative prices are flexible and their time rates of change are increasing linear functions of sectoral excess demands:

$$\dot{p}(t) = -(I-A)q(t) \qquad (2.2)$$

Equations 2.1 and 2.2 represent an example of a dynamic control problem where p(t) are the state variables, q(t) are the control variables, 2.1 is the objective functional and 2.2 are the equations of motion, i.e. the n-dimensional set of constraints on the objective functional.

The Hamiltonian function is

$$H(t,p,q,\psi) = \psi_0 F(t,p,q) - \psi^t(t)(I-A)q(t) = \\ = \psi_0 p^t(t)(I-A)q(t) - \psi^t(t)(I-A)q(t) \qquad (2.3)$$

where $8_0$, $8^t$ (t) are conjugate or costate variables. Necessary conditions for a maximum are

$$\dot{p}(t) = D_{8^t} H = -(I - A)q(t) \tag{2.4}$$

$$\dot{8}(t) = D_p H = 8(t)(I - A)q(t) = 0 \tag{2.5}$$

where $D_{8^t}H$ and $D_pH$ are vectors of first derivatives of H. Hence

$$\dot{8}(t) = -8_0(I - A)q(t) \tag{2.6}$$

. 8 (t) is a scalar multiple of p(t). On the other hand, from equation 2.5 we have

$$D_q H = 8_0 \dot{p}^t(t)(I - A) - 8^t(t)(I - A) = 0 \tag{2.7}$$

hence

$$8^t(t) = \dot{p}^t(t)8_0 \tag{2.8}$$

and, setting $8_0 = 1$

$$8^t(t) = \dot{p}^t(t) \tag{2.9}$$

Necessary conditions tell us that, when the objective functional is maximized with respect to the control variable q(t), shadow prices 8(t) coincide with excess prices. In this case, H(t) is identically 0 at all t, therefore equation 2.6 does not help in determining the optimal path of q(t).

We shall therefore turn our attention to the transversality condition. If the planning horizon is unbounded and the right hand side boundary is free, we can construct a one-parameter family of admissible trajectories of the state variables p(t;ß), with p(t;0)=p(t,q*(t)) where q*(t) is an optimum control; the transversality condition then requires that

$$\lim_{t\to\infty} p^t(t;0)D_{\beta}p(t;0)=0 \qquad (2.10)$$

where $D_{\beta}p(t;0)$ is the derivative of $p(t;\beta)$ at $p(t,q^*(t))$. A solution to 2.10 is

$$\lim_{t\to\infty} p^t(t;0)=0 \qquad (2.11)$$

i.e the transversality condition is satisfied if long period equilibrium is globally asymptotically stable. Since p(t) depends on q(t), equation (2.10) is in fact a constraint on the dynamic behaviour of q(t). From equation 2.4 we have

$$p^t(t)=\int_0^t (I-A)q(s)ds+k=(I-A)\int_0^t q(s)ds+k \qquad (2.12)$$

Setting $k=(y_0-y_0)$, i.e equal to excess final demand at $t=0$, we have

$$(I-A)\lim_{t\to\infty}\left\{\int_0^t q(s)ds+(y_0-y_0)\right\}=0 \qquad (2.13)$$

hence the constraint on the dynamic behaviour of q(t) is

$$\lim_{t\to\infty}\int_0^t q(s)ds=(I-A)^{-1}(y_0-y_0) \qquad (2.14)$$

This means that long-period equilibrium prices are globally asymptotically stable if gross outputs converge asymptotically to <u>any initial level of gross demand</u>. Long period equilibrium prices are therefore independent of output or, equivalently, they are not associated with any particular level of final demand or of employment. On the other hand, the assumption that gross outputs are rapidly adjusted to demand and equation 2.4 necessarily imply that equation 2.10 (the transversality condition) is verified. We might call this assumption the "Keynesian multiplier asymptotic stability assumption".

However, this is not the case in Goodwin's "Walrasian" model. In order to find Goodwin's solution to the optimal control problem, let us first assume that the planning horizon is bounded, i.e. that the time interval over which the objective functional is to be maximized ends at $t_1$, which is finite.

Call $M_1$ the smooth manifold generated by the intersection of s hypersurfaces $S^u(t,p,q)=0$, u=1...s, $s\geq 1$, $M_f$ the manifold generated by equation $t-t_1=0$ and set $M=M_1\cap M_f$. Then, if {p,q} ends at the point $\{t_1,p(t_1),q(t_1)\}\in M$, and $q(t_1)$ is an optimal control, the transversality condition requires that vector {p,q} must be

orthogonal to $M_1(t_1)$ or, equivalently, normal to any vector in the tangent hyperplane at $\{p(t_1),q(t_1)\}$. We shall prove that:

1) the transversality condition is satisfied if $M_1$ is the manifold generated by the bilinear form

$$S(t,p,q)=q^t(t)\boldsymbol{M}^t\boldsymbol{M}\,q(t)+p^t(t)\boldsymbol{M}\,\boldsymbol{M}^t\,p(t)-k=0 \qquad (2.15)$$

where $k\neq 0$ and M is a non-singular, block diagonal matrix

$$M=\begin{pmatrix} B^t & 0 \\ 0 & B \end{pmatrix} \qquad (2.16)$$

2) If B=(I-A) and the time behaviour of prices is given by the equations of motion 2.4, the optimal control is given by

$$q(t)=(I-A^t)p(t) \qquad (2.17)$$

Proof:

Suppose that B=I. Equation 2.15 then becomes

$$S(t,p,q)=q^t(t)q(t)+p^t(t)p(t)-k=0 \qquad (2.18)$$

The equation of a vector $\bar{\gamma}$ in the tangent hyperplane through $\bar{\gamma}=\{p,q\}\in M_1$ is

$$\bar{\gamma}=D_{\gamma}\bar{\gamma}(\gamma-\bar{\gamma}) \qquad (2.19)$$

where $D_{\gamma}\bar{\gamma}$ is the Jacobian matrix of $\gamma$ at $\bar{\gamma}$. The transversality condition is

$$\bar{\gamma}^t(t_1).(t_1)=\bar{\gamma}^t(t_1)D_{\gamma}\bar{\gamma}(t_1)\left(\gamma(t_1)-\bar{\gamma}(t_1)\right)=0 \qquad (2.20)$$

which is always satisfied by any $\bar{>}(t_1) \in M_1(t_1)$[1]. Hence, if B=I, an optimal trajectory which satisfies the transversality condition at $t_1$ is any point $\{\bar{p}(t_1), \bar{q}(t_1)\}$ which is a solution of 2.18. Differentiating 2.18 with respect to time we obtain:

$$2q^t(t)\dot{q}(t) + 2p^t(t)\dot{p}(t) = 0 \qquad (2.21)$$

If we set A=0 in the equations of motion 2.4 and substitute $\dot{p}(t)$ in 2.21, we obtain

hence $$q^t(t)\dot{q}(t) - q^t(t)p(t) = 0 \qquad (2.22)$$

and $$q^t(t)(\dot{q}(t) - p(t)) = 0 \qquad (2.23)$$

$$\dot{q}(t) = p(t) \qquad (2.24)$$

Now, assume that B=(I-A). S becomes

$$^t(t)(I-A)(I-A^t)p(t) + q^t(t)(I-A^t)(I-A)q(t) - \qquad (2.25)$$

Setting $ex>_1 = (I-A^t)p$, $ex>_2 = (I-A)q$ and $ex> = \{ex>_1, ex>_2\}$, proposition (1) can be proved as in the previous case[2]. The transversality condition now becomes

$$t_1) = \{\bar{p}^t(t_1), \bar{q}^t(t_1)\}\mathbf{M}^t D_{ex>} e\bar{x}> \mathbf{M}\{(p(t_1) - \bar{p}(t_1)), (q \qquad (2.26)$$

which is verified by any $\{\bar{p}, \bar{q}\} \in M$. Since $\mathbf{M}^t\ D_{ex>}\ (ex>)\mathbf{M}$ is the image of $D_{p,q}\ (\bar{p}, \bar{q})$ in the space spanned by

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[1]See appendix 2.A.

[2]See appendix 2.B.

$$\boldsymbol{M} = \begin{pmatrix} (I-A^t) & 0 \\ 0 & (I-A) \end{pmatrix} \tag{2.27}$$

2.26 and 2.20 are the same equation in different coordinates. The time derivative of S is now

$$\dot{S}^1(t,p,q) = \dot{p}^t(t)(I-A)(I-A^t)p(t) + 2p^t(t)(I-A)(I-A^t)\dot{p}(t) \tag{2.28}$$

which, together with equation 2.4, implies

$$p^t(t)(I-A)[-p(t)+(I-A^t)p(t)] = 0 \tag{2.29}$$

Hence, the dynamic behaviour of the optimal control is given in this case by

$$q(t) = (I-A^t)p(t) \tag{2.30}$$

which proves proposition (2).

We will now release the assumption that the planning horizon is finite. Our model merely requires that

$$\lim_{t\to\infty} \lambda(t) \tag{2.31}$$

lies in a bounded manifold $M$, therefore, if $\lim_{t\to\infty} \lambda(t)$ exists, it is possible to choose $\lambda(t)$ so that 2.31 is orthogonal to $M$ at the terminal point. When this is done, any necessary conditions at the terminal point of the trajectory will be satisfied[3].

Consider the following smooth hypersurfaces

$$S^1(t,p,q) = p^t(t)(I-A)(I-A^t)p(t) + q^t(t)(I-A^t \tag{2.32}$$

where $k \neq 0$ is a finite real constant, and

[3]See Hadley and Kemp (1971), theorem 4.3.2, p. 246.

$$S^2(t,p,q) = t - t_1 = 0 \qquad (2.33)$$

where $t_1 \geq w$, and $w$ is a sufficiently large real number. Their intersection

$$M = \bigcap_{u=1,2} S^u \qquad (2.34)$$

is a (2n-1) dimensional smooth manifold[4] in $E^*$ x E x $\mathbb{R}$. Equation 2.32 states that both $\lim_{t\to\infty} p(t)$ and $\lim_{t\to\infty} q(t)$ must lie in $M_1$, i.e. in the manifold generated by $S^1$. If $k \neq 0$ is finite, $M_1$ is bounded[5]. From equation 2.4 we know that, since $\lim_{t\to\infty} q(t)$ exists, $\lim_{t\to\infty} \dot{p}(t)$ also exists. The intersection of $S^1$ and $S^2$ is the set of points $\{p(t_1),q(t_1)\}$ which satisfy the bilinear map 2.32 at $t=t_1$, hence $\{p(t_1), q(t_1)\} \in M$, and $\{p(t_1),q(t_1)\}$ is orthogonal to $M_1$ $(t_1)$. This holds for any $t_1 \geq w$, with sufficiently large w, hence $\lim_{t\to\infty} \{p(t),q(t)\}$ is orthogonal to $M$ at the terminal point and the dynamic behaviour of the optimal control is again given by equation 2.30.

We may then conclude that Goodwin's "Walrasian" prices and quantities adjustment equations can be regarded as the solutions to an optimal control problem with an unbounded planning horizon, provided that the terminal point lies in a bounded smooth manifold generated by equation 2.32. This equation states that the the sum of the squared modules of the two n-vectors p(t)(I-A) and (I-A)q(t), which respectively represent excess profits and excess final demands per unit of output, must remain constant during the whole adjustment process; i.e. that all solution curves in the 2n-Euclidean space of excess profits and excess final demands per unit of output lie on a hypersphere of radius $\sqrt{k}$, centred at the origin. In physics, this condition means that the total energy of the

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[4] $E^*$ and E are dual, Euclidean n-spaces and $\mathbb{R}$ is the real line.

[5] See appendix 2.C.

system is conserved[6]. How shall we interpret this condition in the context of a model aimed at representing the process by which a set of n "Walrasian" markets reacts to disequilibrium?

We know that the necessary conditions for a maximum of the objective functional

$$(p'^{t} - p^{*t})(I - A)(q' - q^{*}) \qquad (2.35)$$

where p', p* are actual and equilibrium relative prices respectively, and q', q* are actual and equilibrium gross products respectively, require that

$$\theta(t) = p(t) = p'^{t}(t) + p^{*}(t) \qquad (2.36)$$

i.e. that shadow prices are equal to current market prices. If

$$q' = q^{*} \qquad (2.37)$$

the objective functional is equal to 0. However the terminal condition requires that

$$(p'^{t} - p^{*t})(I - A)(I - A^{t})(p' - p^{*}) = k \neq 0 \qquad (2.38)$$

and the transversality condition

$$\theta(t) = (I - A^{t})p(t) \qquad (2.39)$$

tells us that quantities will change as long as p' ≠ p*. Hence the terminal condition excludes both the possibility that prices and quantities converge asymptotically to their long period values, and the possibility that prices and quantities increase or decrease infinitely.

In standard Cartesian coordinates the origin of axes represents long period equilibrium. Hence √k - the distance from the origin of vector {(I-A)q, (I-$A^{t}$ )p } - can be interpreted as a measure of the "degree of disequilibrium" which exists in the economic system. It then becomes evident that the terminal condition 2.15, by keeping

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[6]See Hirsch and Smale (1974), p. 292.

the existing degree of disequilibrium constant, serves the purpose of assuring that the model is globally **stable**, although it is not globally **asymptotically stable**[7]. We might then call the terminal condition 2.15 "Goodwin's Walrasian stability assumption". On the other hand, as we shall soon see, Goodwin's Hamiltonian model is **structurally unstable**[8].

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[7]Of course, this condition constrains the dynamic behaviour of the model rather arbitrarily. I leave it to the reader to decide whether it is more or less legitimate than the previously mentioned "Keynesian multiplier stability assumption" or the local asymptotic stability assumption, which is a common feature of a large part of the recent new-classical literature on economic growth. On this point see, for example, Serena Sordi (1990).

[8]i.e., the unperturbed and the perturbed vector fields are not topologically equivalent. See Guckenheimer and Holmes (1990), p. 39.

## Appendix 2.A

Set

$$M=\left\{(\bar{p},\bar{q})\in R^{2n}\,\middle|\,k-\sum_{i=2}^{n}\bar{p}_i^2-\sum_{i=1}^{n}\bar{q}_i^2=\bar{p}_1^2\right\} \tag{2.A.1}$$

The Jacobian matrix of $\bar{>}$ is

$$D_{\bar{>}}=\begin{pmatrix}\frac{\bar{p}_2}{*\bar{>}_1} & \cdots & \frac{\bar{p}_n}{*\bar{>}_1} & \frac{\bar{q}_1}{*\bar{>}_1} & \cdots & \frac{\bar{q}_n}{*\bar{>}_1}\\ 0 & & & & I_{2n-1} & \end{pmatrix} \tag{2.A.2}$$

where

$$\frac{1}{*\bar{>}_j}\,\middle|\,j=2\ldots n \tag{2.A.3}$$

The transversality condition at $\bar{q}=\bar{q}_1$ is

$$\bar{>}^t D_{\bar{>}}(\bar{>}-\bar{>})=0 \tag{2.A.4}$$

where

$$\cdot_1=-\frac{1}{\bar{p}_1}\left(\sum_{i=2}^{n}p_i\bar{p}_i-\sum_{i=2}^{n}\bar{p}_i^2+\sum_{i=1}^{n}q_i\bar{q}_i-\sum_{i=1}^{n}\bar{q}_i^2\right) \tag{2.A.5}$$

Since

$$\sum_{i=2}^{n}\bar{p}_i^2+\sum_{i=1}^{n}\bar{q}_i^2+\bar{p}_1^2=k \tag{2.A.6}$$

we have

$$\cdot_1=-\frac{1}{\bar{p}_1}\left(\sum_{i=2}^{n}p_i\bar{p}_i+\sum_{i=1}^{n}q_i\bar{q}_i-k+\bar{p}_1^2\right) \tag{2.A.7}$$

and

$$[\cdot_i]=[>_i-\bar{>}_i],\ i=2\ldots 2n \tag{2.A.8}$$

Hence

$$\begin{aligned}(\bar{>}^t\cdot)&=-\left(\sum_{i=2}^{n}p_i\bar{p}^i+\sum_{i=1}^{n}q_i\bar{q}_i-k+\bar{p}_1^2\right)+\\&+\left(\sum_{1=2}^{n}p_i\bar{p}_i^2+\sum_{i=1}^{n}q_i\bar{q}_i^2-\sum_{i=2}^{n}\bar{p}_i^2-\sum_{i=1}^{n}\bar{q}_i^2\right)=0\end{aligned} \tag{2.A.9}$$

## Appendix 2.B

Set

$$ex> = \begin{pmatrix} ex>_1 \\ ex>_2 \end{pmatrix} = \boldsymbol{M} \begin{pmatrix} p \\ q \end{pmatrix} \qquad (2.B.1)$$

Then

$$;^1(ex>_1, ex>_2) = \sum_{i=1}^{n} (ex>_{1i})^2 + \sum_{i=1}^{n} (ex>_{2i})^2 - k = \qquad (2.B.2)$$

where $p^t (I-A^t)\left(\begin{array}{c} 0 \\ \hline *\overline{ex>}_{11} \\ \hline *\overline{ex>}_{1j} \end{array} \middle| j=2\ldots n\right) + q^t (I-A^t)\left(\begin{array}{c} *\overline{ex>}_{11} \\ \hline *\overline{ex>}_{2j} \end{array} \middle| j=1\ldots n\right) = k = 0$ (2.B.3)

$$x> \left(\begin{array}{c} 0 \\ \hline *\overline{ex>}_{11} \\ \hline *\overline{ex>}_{1j} \end{array} \middle| j=2\ldots n\right) = I_{(2n-1)} \left(\begin{array}{c} \overline{ex>}_{1j} \\ \hline \overline{ex>}_{11} \end{array} \middle| j=2\ldots n\right) \qquad (2.B.4)$$

$$\left(\begin{array}{c} *\overline{ex>}_{11} \\ \hline *\overline{ex>}_{2j} \end{array} \middle| j=1\ldots n\right) = -\left(\begin{array}{c} \overline{ex>}_{2j} \\ \hline \overline{ex>}_{11} \end{array} \middle| j=1\ldots n\right) \qquad (2.B.5)$$

and

$$\sum_{j=2}^{n} (\overline{ex>}_{1j})^2 + \sum_{j=1}^{n} (\overline{ex>}_{2j})^2 = k - (\overline{ex>}_{11})^2 \qquad (2.B.6)$$

$$\overline{ex>}_{11} = \sqrt{k - \sum_{j=2}^{n} (\overline{ex>}_{1j})^2 - \sum_{j=1}^{n} (\overline{ex>}_{2j})^2} \qquad (2.B.7)$$

The transversality condition then becomes

Setting

$$(\overline{ex>})^t . = (\overline{ex>})^t D_{\overline{ex>}} \overline{ex>} (ex>-\overline{ex>}) = \{\overline{p}^t (I$$

$$. \left(\begin{array}{c} 0 \\ \hline \overline{ex>}_{11} \\ \hline \overline{ex>}_{1j} \end{array} \middle| j=2\ldots n\right) (1-a_{11}) p_1 - \sum_{j=2}^{n} \left(\begin{array}{c} \overline{ex>}_{2j} \\ \hline \overline{ex>}_{11} \end{array} \middle| j=1\ldots n\right) a_{1j} p_j \qquad (2.B.8)$$

$$0 \qquad = \sqrt{k - \sum_{j=2}^{n} (\overline{ex>}_{1j})^2 - \sum_{j=1}^{n} (\overline{ex>}_{2j})^2} \qquad (2.B.9)$$

we have

where $D_{\overline{ex>}} = \begin{pmatrix} 0 & -\frac{1}{\overline{ex>}_{11}}\left\{\bar{p}^{t}_{(n-1)}(I-A)_{(n-1)},\ \bar{q}^{t}(I-A^{t})\right\} \\ \bar{0}^{t}_{n-1}(I-A)_{(n-1)} & I_{(2n-1)} \end{pmatrix}$ (2.B.10)

and $\bar{0}^{t}_{n-1}(I-A)_{(n-1)} = (\overline{ex>}_{12}, \ldots, \overline{ex>}_{1n})$ (2.B.11)

hence $D_{\langle ex\rangle}\overline{\langle ex\rangle}(\langle ex\rangle - \overline{\langle ex\rangle}) =$

$$k + \left(\frac{1}{\overline{\langle ex\rangle}_{11}}\right)\left[-k_t + (\overline{\langle ex\rangle}_{11})^2 + \bar{p}^t_{n-1}(I-A)_{(n-1)}(I-A^t)_{(n-1)}p_{(n-1)} + \bar{q}^t(I-A^t)_{(n-1)}p_{(n-1)} + \overline{\langle ex\rangle}_{11}\bar{p}^t_{(n-1)}(I-A)_{(n-1)}(I-A^t)_{(n-1)}p_{(n-1)} + \bar{q}^t(I-A^t)_{(n-1)}\right] \tag{2.B.13}$$

$$= \begin{pmatrix} \langle ex\rangle_{12} - \overline{\langle ex\rangle}_{12} \\ \vdots \\ \langle ex\rangle_{2n} - \overline{\langle ex\rangle}_{2n} \end{pmatrix} \tag{2.B.12}$$

## Appendix 2.C

Equation 2.32 can be reformulated as follows

$$S(t,p,\theta)=\sum_{i=1}^{2n}(\langle ex\rangle_i)^2\le k \qquad (2.C.1)$$

where k is any finite real constant $\neq 0$. If $\langle ex\rangle_j=0$, $j=1...i-1, i+1...2n$, 2.C.1 implies that

$$\langle ex\rangle_i=\pm\sqrt{k} \qquad (2.C.2)$$

Since this holds for any $i=1...2n$, $+\sqrt{k}$, $-\sqrt{k}$ are the upper and lower bounds of each $\langle ex\rangle_i(t)\in\langle ex\rangle(t)$. Hence $\langle ex\rangle(t)$ is bounded. But

$$\langle ex\rangle(t)\in M_1 \qquad (2.C.3)$$

hence $M_1$ is bounded.

3. **A symplectic change of coordinates.**

Let us call ex $q_i$ the i-th element of vector (I-A)q, equal to the excess demand of the i-th commodity, and ex $p_i$ the i-th element of vector $(I-A^t)p$, equal to the excess profit of the i-th sector. Hamilton's equations 2.4 and 2.30 can be written as

$$\dot{>} = \begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} -(I-A)q \\ (I-A^t)p \end{pmatrix} = J \begin{pmatrix} exp \\ exq \end{pmatrix} = Jex> \qquad (3.1)$$

where

$$J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \qquad (3.2)$$

DEFINITION 3.I: A *canonical* or *symplectic* coordinates transformation of 3.1 is a transformation 0=f(>), where f: $\mathbb{R}^n$ x $\mathbb{R}^n$ → $\mathbb{R}^n$ x $\mathbb{R}^n$ is smooth, which satisfies the following conditions:

a) if >(t) is the solution of Hamilton's equations 3.1., 0(t)=f(>(t)) is the solution of equations

$$\dot{0} = Q\dot{>} = QJ\,grad_{ex>}k(ex>) = QJQ^t\,grad_{ex0}k(ex>(ex0)) \qquad (3.I.1)$$

where $Q = \left[\frac{\partial 0^i}{\partial >^j}\right]_{ex>}$ is the Jacobian of f and $Q^t$ is the transpose of Q.

b) if and only if $QJQ^t=J$, the equations for 0 will be Hamiltonian with energy[9]

$$h(ex0) = k(ex>(ex0)) \qquad (3.I.2)$$

In order to simplify argument, let us make the following

HYPOTHESIS 3.i : (I-A) has n distinct real eigenvalues $\theta_1, \ldots, \theta_n$.

^

[9]See Abraham and Marsden (1987) pp. XXI, XXII.

Call 8 the diagonal matrix of the eigenvalues of (I-A) and X the matrix of the eigenvectors belonging to them. Then we have

$$X(I-A)X^{-1}=\hat{8}=(X^{-1})^t(I-A^t)X^t \tag{3.3}$$

We shall prove that the transformation

$$Q=Q>=\begin{pmatrix}(X^{-1})^t & 0\\ 0 & X\end{pmatrix}\begin{pmatrix}p\\ q\end{pmatrix}=\begin{pmatrix}P\\ x\end{pmatrix} \tag{3.4}$$

is a symplectic trasformation, i.e. it satisfies conditions a) and b) of Definition 3.1. Condition b) can be easily checked, since

Condition a) is satisfied if

$$=\begin{pmatrix}(X^{-1})^t & 0\\ 0 & X\end{pmatrix}\begin{pmatrix}0 & -I\\ I & 0\end{pmatrix}\begin{pmatrix}X^{-1} & 0\\ 0 & X^t\end{pmatrix}=\begin{pmatrix}0 & -I\\ I & 0\end{pmatrix}=J \tag{3.5}$$

$$0=Q8=Q1\nabla_{ex>}\,k(ex>)=J\,\nabla_{ex0}\,h(ex0) \tag{3.6}$$

From 3.4 we have

Setting

$$ex0=QMQ^{-1}Q>=\begin{pmatrix}\hat{8} & 0\\ 0 & \hat{8}\end{pmatrix}\begin{pmatrix}B\\ x\end{pmatrix}=\begin{pmatrix}\hat{8}B\\ \hat{8}x\end{pmatrix} \tag{3.7}$$

$$k(ex>)=k(ex>(ex0))=$$

from condition b) we have

$$=\frac{1}{2}(ex>)^t(ex>)=\frac{1}{2}(ex>(ex0))^t(ex>(ex0)) \tag{3.8}$$

$$(ex>)^t(ex>)=(ex0)^t(ex0) \tag{3.9}$$

Since the column vector ex> is trasformed contravariantly, the scalar product k(ex>) is invariant to the coordinates transformation 3.7 only if the row vector ex> is transformed covariantly[10], i.e. if[11]

$$(ex0)^t=(ex>)^tQ^{-1} \tag{3.10}$$

Thus equation 3.8 becomes

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[10]See, for example, Eisele and Mason (1970), part II.

[11] i.e. $(ex>)^t = (ex0)^t\ Q$.

$$(\nabla_{ex0}(ex0))=(ex>(ex0))^t(ex>(ex0))=(ex0)^tQQ^{-1}(e \quad (3.11)$$

and

Substituting expression (3.12) in equation 3.6 we obtain

$$\nabla_{ex0}k(ex>(ex0))=\left[\frac{*k(ex>(ex0))}{*ex0}\right]^t=Q^tQ(ex0)= \quad (3.12)$$

$$\dot{0}=Q\dot{0}=QJ\nabla_{ex>}k(ex>)=QJ[(ex>)^t]^t=QJ[(ex0)^tQ]^t= \quad (3.13)$$

Hence Hamilton's equations in the new symplectic coordinates are

$$QJQ^tex0=QJQ^t\nabla_{ex0}k(ex>(ex0))=Jex0$$

$$\dot{0}=\hat{8}x \quad (3.14)$$

as in Goodwin and Punzo (1987), pp. 79-82.

$$\dot{x}0=\hat{8}B$$

Alternatively, condition a) can be proved as follows: set

$$\dot{0}=J\nabla_{ex>}k(ex>) \quad (3.15)$$

hence

$$Q\dot{0}=QJ\nabla_{ex>}k(ex>) \quad (3.16)$$

From

$$ex0 = Q\,ex> \quad (3.17)$$

we have

$$\frac{*0}{*>}=\frac{*(ex0)}{*(ex>)}=Q \quad (3.18)$$

Since $QJQ^t=J$ and 3.9 is verified, we have

$$\frac{*k(ex>)}{*ex>} = \frac{*k(ex>(ex0))}{*ex0}\cdot\frac{*ex0}{*ex>} \quad (3.19)$$

and

$$\nabla_{ex>}k(ex>(ex0)) = \left[\frac{*k(ex>)(ex0))}{*ex0}\cdot\frac{*ex0}{*ex>}\right]^t = \quad (3.20)$$

Hence 3.16 becomes

$$=\left(\frac{*ex0}{*ex>}\right)^t\nabla_{ex0}k(ex>(ex0))=Q^t\nabla_{ex0}h(ex0) \quad (3.21)$$

$$\dot{0}=QJQ^tgrad_{ex0}h(ex0)$$

Integral curves of system 3.14 can be easily found : taking the time derivative of the first equation of 3.14 and substituting $\dot{0}$ from the second equation we obtain a system of n second order linear differential equations

$$\ddot{B} = -\hat{\omega}^2 B \qquad (3.22)$$

which, under hypothesis 3.i, has only imaginary roots with zero real parts

$$\mu_j = \pm\sqrt{-\omega_j^2} = \pm i\omega_j \qquad (3.23)$$

By Euler's formula

$$e^{\pm i\omega_j t} = \cos\omega_j t \pm i \sin\omega_j t \qquad (3.25)$$

hence, in symplectic coordinates, all integral curves of the Hamiltonian system 3.1 lie on a manifold given by the Cartesian product of n circles of radius 1. In Goodwin's own words: "The motion is dynamically stable, in the sense of bounded, but is not asymptotically stable towards equilibrium. It is structurally unstable (...) so that a slight error in, or perturbance of, the parameters would lead either to the disappearance of the cycle or to its explosion without limit."[12] Figure 1 is the phase portrait of a numerical example of system 3.14 (Navajo), in the case where n=2. Figure 2 is the graph of B vs. time. The parameters and the initial conditions of this simple two-sector system are given in table 1. This numerical example provides the foundations on which the more complex examples of the following paragraphs will be built.

In order to have a greater insight into the dynamic behaviour of the vector field 3.1 and to simplify the analysis of the perturbed Hamiltonian, which will be introduced in the next paragraph, it is useful to perform a further, symplectic change of coordinates. For the moment, we will go on assuming that n=2; equations 3.14 can then be written as

$$\dot{\theta}_1 = -\omega_1 x_1 \qquad (3.26.1)$$

$$\dot{x}_1 = \omega_1 \theta_1 \qquad (3.26.2)$$

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[12] Goodwin and Punzo, (1987), p. 73.

$$\dot{B}_2 = -\omega_2 x_2 \qquad (3.26.3)$$
$$\dot{x}_2 = \omega_2 B_2 \qquad (3.26.4)$$

and equation 3.22 can be written as

$$\ddot{B}_1 = -\omega_1^2 B_1 \qquad (3.27.1)$$
$$\ddot{B}_2 = -\omega_2^2 B_2 \qquad (3.27.2)$$

Equations 3.26 and 3.27 make clear that vector field 3.14 can be divided into two uncoupled Hamiltonians:

$$k_1 = \frac{1}{2}(\omega_1^2 B_1^2 + \omega_1^2 x_2^2) \qquad (3.28.1)$$
$$k_2 = \frac{1}{2}(\omega_2^2 B_2^2 + \omega_2^2 x_2^2) \qquad (3.28.2)$$

Setting

$$v_1 = \omega_1 x_2 \qquad (3.29.1)$$

we have

$$k_1 = \frac{1}{2}(\omega_1^2 B_1^2 + v_1^2) \qquad (3.30.1)$$
$$k_2 = \frac{1}{2}(\omega_2^2 B_2^2 + v_2^2) \qquad (3.30.2)$$

In this case it is possible to further transform the linear non-orthogonal coordinates of system 3.14 into non-linear polar coordinates (action angle coordidates[13]). Set

[13]See Guckenheimer and Holmes (1983), pp.212-215.

hence $$B_2 = \sqrt{\frac{2I}{\omega_2}}\sin N \qquad (3.31)$$

The Hamiltonian becomes $$v_2 = \sqrt{2I\omega_2}\cos N \qquad (3.32)$$

hence $$k = \omega_2 I + \frac{\omega_1^2 B_1^2 + v_1^2}{2} = k(I, N, B_1, v_1) \qquad (3.33)$$

Setting $$I = \frac{1}{\omega_2}\left(k - \frac{\omega_1^2 B_1^2 + v_1^2}{2}\right) = I(k, N, B_1, v_1) \qquad (3.34)$$

the reduced Hamiltonian system becomes:

$$B_1' = \frac{dB_1}{dN} = v_1 \qquad (3.35)$$

$$v_1' = \frac{dv_1}{dN} = -\omega_1^2 B_1 + \frac{\omega_1}{\omega_2}\left(k - \frac{\omega_1^2 B_1^2 + v_1^2}{2}\right)$$

Since k does not depend explicitly on N, the reduced system 3.36 is autonomous. The linearized Poincaré map can thus be easily obtained by solving system 3.36 based at $\{B_1^\circ, v_1^\circ\}$. The general solution is

$$B_1(N) = B_1^\circ \cos \omega N + \frac{v_1^\circ}{\omega_1}\sin \omega N \qquad (3.37)$$

where $\omega = \omega_1/\omega_2$ is 2π periodic in N. Thus the linearized Poincaré map is

$$v_1(N) = -B_1^\circ \omega_1 \sin \omega N + v_1^\circ \cos \omega N$$

$$P_k^\circ = \begin{pmatrix} \cos 2\pi\omega & \frac{1}{\omega_1}\sin 2\pi\omega \\ -\omega_1 \sin 2\pi\omega & \cos 2\pi\omega \end{pmatrix} \begin{pmatrix} B_1 \\ v_1 \end{pmatrix} \qquad (3.38)$$

Its eigenvalues are again complex conjugate, with unit modules

$$\mu_{1,2} = \cos 2\pi\omega \pm i \sin 2\pi\omega \qquad (3.39)$$

By applying De Moivre's theorem we obtain the solution for $B_1$,

$$B_1(t) = B_1^\circ(\cos 2\pi\omega t - \phi) \qquad (3.40)$$

which is periodic with period $2\pi\omega$. Thus the long period equilibrium solution {0,0} is "an elliptic centre surrounded by a family of closed curves filled with periodic points if S is rational, and

with dense orbits if S is irrational"[14]. In our numerical example S is an irrational number, hence flows are dense orbits, as can be easily seen from figures 1 and 2.

Also variables $<_1$ and $B_1$ can be transformed into a second set of action angle variables. Set

$$B_1 = \sqrt{\frac{2J}{8}} \sin R \tag{3.41}$$

and

$$v_1 = \sqrt{2J8_1} \cos R \tag{3.42}$$

The second Hamiltonian becomes

$$k_1 = 8_1 J \tag{3.43}$$

and its vector field is

$$\begin{aligned} J^{\cdot} &= -\frac{*k_1}{*R} = 0 \\ R^{\cdot} &= \frac{*k_1}{*J} = 8_1 \end{aligned} \tag{3.44}$$

[14]See Guckenheimer and Holmes (1983) p. 215.

The complete dynamic system is then reduced to

$$\dot{J}=0;\ \dot{R}=\theta_1;\ \dot{I}=0;\ \dot{N}=\theta_2 \qquad (3.45)$$

which, for initial conditions $\{J^\circ, I^\circ, R^\circ, N^\circ\}$, has the solution

$$J=J^\circ \qquad I=I^\circ$$
$$R(t)=\theta_1 t+R^\circ \qquad N(t)=\theta_2 t+N^\circ \qquad (3.46)$$

and, in terms of the original variables

$$x_1(t)=\sqrt{\frac{2I}{\theta_1}}\sin(\theta_1 t+R^\circ) \qquad x_2(t)=\sqrt{\frac{2I}{\theta_1}}\cos(\theta_1 t+R^\circ)$$
$$x_3(t)=\sqrt{\frac{2I}{\theta_2}}\sin(\theta_2 t+N^\circ) \qquad x_4(t)=\sqrt{\frac{2I}{\theta_2}}\cos(\theta_2 t+N^\circ) \qquad (3.47)$$

"Thus the four dimensional phase space is filled with two-dimensional tori[15], given by $J=J^\circ$, $I=I^\circ$ and each torus carries rational or irrational flows depending on the ratio $\theta_1/\theta_2$. In general, n-degree of freedom integrable Hamiltonian systems give rise to flows on n-dimensional tori"[16]. In our case the j-th vector field can be written as

$$\begin{pmatrix}\dot{R}_j\\ \dot{\theta}_j\end{pmatrix}=\begin{pmatrix}0 & \theta_j\\ 0 & 0\end{pmatrix}\begin{pmatrix}R_j\\ x_j\end{pmatrix} \qquad (3.48)$$

In the two sector case (equation 3.45) the linearized Poincaré map is degenerated[17]. Hence, the Kolmogorov-Arnold-Moser theorem[18], which asserts that, if the period of R is a function of J, most of the closed irrational orbits of the unperturbed Poincaré map are preserved for sufficiently small perturbations, cannot be applied. The system is structurally unstable.

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[15] $S^1 \times S^1$ is given by the Cartesian product of two circular phase spaces $S^1 \times S^1$.

[16] Guckenheimer and Holmes (1983) p. 59.

[17] In the two-dimensional example we can set $\dot{R}=\theta_1$. The non degeneracy condition requires that $\theta'(J)\neq 0$. Since, in this case, $\dot{R}$ is a real constant, this condition is violated. (Guckenheimer and Holmes, 1983, p.219).

[18] See Guckenheimer and Holmes (1983), p. 219.

## 4. **Consumer demand functions**.

In this paragraph we will add a slightly non-linear perturbation to the linear dynamic adjustment model analysed in paragraphs 2 -3. In the first chapter of Chaotic Economic Dynamics Goodwin assumes that only wage earners spend for consumption, that wage income is entirely spent, that the uniform, nominal wage rate w remains constant during the whole adjustment process[19], and that consumers expenditure is distributed among different goods and services in fixed proportions. Under these assumptions, the Marshallian demand function for the j-th consumption good is given by

$$q^d_j = \frac{s_j}{p_j} w l' \qquad (4.1)$$

where $q^d_j$ is the demand of good j, $p'_j$ is its price, l' is total employment and $s_j$ is the proportion of total wage income spent in purchasing good j. Of course

$$\sum s_j = 1 \qquad (4.2)$$

As is well known, constant expenditure shares are a feature of Cobb-Douglas utility functions. Consumer demand functions like 4.1 can be easily derived from a constrained maximum problem (consumers maximize their current utility under the constraint of a given nominal income), provided the utility function is Cobb-Douglas[20]. On the other hand, demand functions like 4.1 imply that the real wage income is flexible and inversely related to prices.

To investigate some of the possible effects of the adoption of these behavioural hypotheses on the dynamic process analysed in the previous paragraphs, we will restate them as follows:

HYPOTHESIS 4.i The nominal wage per unit of uniform labour is

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[19]In this book Goodwin represents the adjustment process of prices and quantities by a gradient vector field.

[20]See, for example, Varian (1984) pp.128-129.

constant and equal to 1; i.e.

$$w = 1 \tag{4.3}$$

HYPOTHESIS 4.ii Current prices are expressed in index numbers, based at their respective long period equilibrium values; i.e.

$$p_j = p_j' = 1 \tag{4.4}$$

HYPOTHESIS 4.iii The wage is entirely spent for consumption in fixed proportions; i.e.

$$w = \sum_i s_i \tag{4.5}$$

Finally, we will assume that n=2.

From hypotheses 4.ii and 4.iii we have that $s_1$ and $(1-s_1)$ represent equilibrium real consumption per unit of labour, i.e equilibrium commodity wage. Hence current commodity wage is:

$$\mathrm{F} = \begin{pmatrix} \mathrm{F}_1 \\ \mathrm{F}_2 \end{pmatrix} = \begin{pmatrix} \dfrac{s_1}{1+p_1} \\ \dfrac{1-s_1}{1+p_2} \end{pmatrix} \tag{4.6}$$

Define as y*(p*) the two-dimensional vector of equilibrium final demands. We know that

$$q^* = (I - A - sa_l)^{-1} y^* \tag{4.7}$$

and therefore

$$l^* = a_l q^* \tag{4.8}$$

where $a_l$ is the row vector of constant labour inputs coefficients. Excess supply is now given by

$$(I - A)q - [l' + sl^*] = (I - A)q - l(I + \hat{p})^{-1} s + l^* \hat{p}(I + \hat{p})^{-1} s \tag{4.9}$$

where $l = (l' - l^*) = a_l(q' - q^*)$ is current excess employment,

and

$$\hat{p} = Diag(p) \tag{4.10}$$

The dimension of the phase space is now (2n+1)=5, and the vector field is now

$$\dot{p}_i = \frac{s_i}{1 + p_i} \tag{4.11}$$

In symplectic coordinates 4.12 becomes

$$\begin{aligned} \dot{Q} &= (I - A^t)p \\ \dot{P} &= a_l Q = a_l (I - A^t)p \end{aligned} \tag{4.12}$$

Although the number of sectors is only two, the number of parameters involved is relatively high (eight), hence a wide range of dynamic behaviour and bifurcations becomes possible. This makes the stability analysis of the flows on the five dimensional phase space rather difficult. Analysis will therefore be restricted to a brief discussion of the outcomes of the simple two-sector numerical example presented in table 1 (Sioux). The phase portait of this model is given in figure 3. Figures 4, 5 and 6 are the graphs of $B_1$, $B_2$, and l', i.e. of the deviations of current prices from their equilibrium values and of total employment respectively, vs. time.

Since the deviations of current prices from their equilibrium

values tend to vanish and the same holds for l, equilibrium is, in this example, locally asymptotically stable. However, it may be asked whether the zero solution of this model can be regarded as a long period equilibrium. Total employment (figure 5) does **not** tend to an exogenously given constant level, but to an endogenously determined value, which depends on both the behavioural hypotheses on which the model is built and the initial conditions. This means that excess gross products q tend to vanish, i.e that current gross outputs adjust to their respective equilibrium values which, however, are not independent of the adjustment process itself.

This result is, of course, a consequence of the assumption of a linear technology. In a linear production model, long period equilibrium prices are unequivocally defined once technology and income distribution are defined. The same holds for the structure of gross outputs, once technology and the structure of final demand is given, as is the case in Goodwin's 1990 model. However, the absolute level of outputs in each sector (the module of the gross outputs vector) remains indeterminate. Hence, in a dynamic adjustment model like the one presented in this paragraph, **any** level of gross production can be regarded as an equilibrium level, provided it is stationary.

## 5. **Further analysis of the dynamics of a two degree of freedom Hamiltonian**.

The aim of this paragraph is only to provide an example of the interesting potential results which the Hamiltonian approach may lead to, if the assumptions of real wage flexibility and constant returns to scale are released. For this purpose, some rather substantial modifications of Goodwin's basis Hamiltonian model need to be introduced. Formally, these will take the shape of modifications of both the non-linear perturbation introduced in paragaph 4 and the linear unperturbed Hamiltonian analysed in paragraphs 2 and 3.

Firstly, we will assume that the commodity wage $F$ changes if prices change, but these changes will not be sufficient to entirely cancel the initial "degree of disequilibrium" affecting income distribution. As in the case of prices, the "degree of disequilibrium" is measured by the length of vector

$$Fl'-sl^*=l(I+\hat{p})^{-1}s-l^*\hat{p}(I+\hat{p})^{-1}s \tag{5.1}$$

or, in symplectic coordinates

$$[I+Diag(X^t\mathrm{B})]^{-1}s-lX[Diag(X^t\mathrm{B})][I+Diag(X^t\mathrm{B})] \tag{5.2}$$

The perturbation then becomes

$$H_1=\frac{g}{2}F^t[l^*Diag(\mathrm{B}X)X^t-lX^t][l^*X\,Diag(\mathrm{B}X)-lX] \tag{5.3}$$

where

$$0\le g\le 1 \tag{5.4}$$

The two-matrix of right eigenvectors of (I-A) can be written in the form

Setting

$$X=\begin{pmatrix}1 & b_{12}\\ b_{21} & 1\end{pmatrix} \tag{5.5}$$

the perturbed Hamiltonian can now be written as

$$a = s_1^2(1+b_{21}^2) \qquad (5.6)$$
$$b = s_1(1-s_1)(b_{12}+b_{21})$$
$$c = (1-s_1)^2(1+b_{12}^2)$$

where

$$k^g = \frac{1}{2}[\delta^2_1(B^2_1+x^2_1)+\delta^2_2(B^2_2+x^2_2)]+ \qquad (5.7)$$
$$+g\left\{\frac{l^{*2}h}{2}+\frac{1}{2h}\left[\left(\frac{l'\sqrt{a}}{p_1}\right)^2+2b\left(\frac{l'\sqrt{c}}{p_2}\right)^2\right]-\right. \qquad (5.8)$$
$$\left.-\frac{al^*l'}{p_1}-\frac{cl^*l'}{p_2}-bl'\left(\frac{l^*}{p_2}+\frac{l^*}{p_1}-\frac{l'}{p_1p_2}\right)\right\}=k^\circ+gk^1$$

and

$$p_1' = 1+B_1+b_{21}B_2 \qquad (5.9)$$
$$p_2' = 1+b_{21}B_1+B_2$$

$$l' = x+l^* \qquad (5.10)$$

From equation 5.1 one can see that the "degree of disequilibrium" affecting income distribution depends on the deviations of current prices from their equilibrium values and on the deviations of total employment from its equilibrium level l*. Since the nominal wage is constant,

$$(p'-p^*)(I-A) \qquad (5.11)$$

can still be used as a satisfactory measure of excess profits. On the other hand, excess final goods supply

$$(I-A)(q'-q^*) \qquad (5.12)$$

can still be used as a measure of excess supply. Thus, in symplectic coordinates, the Hamiltonian vector field can be written as

$$\dot{B} = -\nabla_{ex\,x}k^\circ(ex\,B,ex\,x)-g\nabla_{ex\,x}k^1(ex\,B,ex\,x)$$
$$\dot{x} = \nabla_{ex\,B}k^\circ(ex\,B,ex\,x)+g\nabla_{ex\,B}k^1(ex\,B,ex\,x) \qquad (5.13)$$

Setting $l^*=1$ and $g=1$, system 5.13 becomes

$$\dot{B} = -\hat{\delta}x+\hat{\delta}^{-1}\left[\frac{t}{t}\right]\left[\frac{t}{t}\right]\left[\frac{t}{t}\right]$$
$$\dot{x} = \hat{\delta}B+\hat{\delta}^{-1}\left[\frac{t}{t}\right]\left[\frac{t}{t}\right]\left[\frac{t}{t}\right] \qquad (5.14)$$

Since $k^\circ=0$, a third conserved quantity has been added to $k_1$ and $k_2$. However, integral curves are still flows on a two-dimensional manifold. A numerical example of this model (Paiute) is also presented in table 1; its phase portrait is presented in figure 7.

Figures 8, 9 and 10 are the graphs of $B_1$, $B_2$ and l' vs. time.

In order to simplify the discussion of the second substantial modification of the basis Hamiltonian system, I assume, for the moment, that ,=0. The hypotheses that the time derivatives of sectoral outputs are linear increasing functions of sectoral excess profits and that the time derivatives of prices are linear increasing functions of sectoral excess demands are now replaced by the assumptions that:

- the changes of the gross output of each sector depend linearly **and** non-linearly on excess profits in **all** sectors;
- the changes of the price of each product depend linearly on excess demands in **all** sectors.

Moreover, as far as gross outputs are concerned, it is assumed that, if positive (negative) deviations from equilibrium are small, all factors of production can be used more (less) efficiently; therefore, returns to scale are increasing (decreasing). However, if the absolute value of deviations from equilibrium exceed a given maximum, the absolute values of efficiency increases (decreases) are progressively reduced to zero. A simple way of stating these hypotheses is the following:

$$\begin{aligned} \theta_1 &= D_{11} \sin f_1(p_1,p_2) + D_{12} f_2(p_1,p_2) \\ \theta_2 &= D_{21} \sin f_1(p_1,p_2) + D_{22} f_2(p_1,p_2) \end{aligned} \qquad (5.15)$$

To minimize the difficulty of analysing the dynamic behaviour of the nonlinear production model, it is useful to assume also that

and that $$C = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix} = X^t \qquad (5.16)$$

$$f_1(p_1,p_2) = B_1$$

Thus, equations 5.15 in symplectic coordinates become: $$f_2(p_1,p_2) = 8_2 B_2 \qquad (5.17)$$

A similar set of simplifying assumptions can be made for the time derivatives of prices. Assume that

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = X\dot{\theta} = (X^{-1})^t \chi_t \begin{pmatrix} \sin B_1 \\ \theta_2 B_2 \end{pmatrix} \tag{5.18}$$

and that

$$\begin{aligned} \dot{\theta}_1 &= L_{11} g_1(q_1, q_2) + L_{12} g_2(q_1, q_2) \\ \dot{\theta}_2 &= L_{21} g_1(q_1, q_2) + L_{22} g_2(q_1, q_2) \end{aligned} \tag{5.19}$$

$$K = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} = X^{-1} \tag{5.20}$$

Thus, in symplectic coordinates, system 5.19 becomes

$$\begin{aligned} g_1(q_1, q_2) &= -x_1 \\ g_2(q_1, q_2) &= -\theta_2 x_2 \end{aligned} \tag{5.21}$$

and the unperturbed Hamiltonian vector field is

$$\begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{pmatrix} = (X^{-1})^t \dot{\theta} = -X X^{-1} \begin{pmatrix} x_1 \\ \theta_2 x_2 \end{pmatrix} = \begin{pmatrix} -x_1 \\ -\theta_2 x_2 \end{pmatrix} \tag{5.22}$$

$$\begin{aligned} \dot{\theta}_1 &= -x_1 \\ \dot{\theta}_2 &= -\theta_2 x_2 \end{aligned} \tag{5.23.1}$$

As in paragraph 2, $B_1$ and $x_2$ can be transformed to action-angle variables: (5.23.2)

$$\dot{\theta}_2 = \theta_2 x_2$$

hence, the final expression of the unpertrubed Hamiltonian is

$$B_2 = \sqrt{\frac{2I}{\theta_2}} \sin N \tag{5.24}$$

$$x_1 = \sqrt{2I\theta_2} \cos N$$

If $k^\circ = 0$, the perturbed Hamiltonian becomes:

$$k = \frac{x_1^2}{2} + (1 - \cos B_1) + \theta_2 I = F(B_1, x_1) + G(I) \tag{5.25}$$

$$k = F(B_1, x_1) + G(I) + \varepsilon k^1(B_1, x_1, N, I) \tag{5.26}$$

$F(B_1, x_1)$ is the Hamiltonian of a simple pendulum with energy

$$k = F(B_1, x_1) \tag{5.27}$$

In a cylindrical phase space:

and $$F(B_1, x_1): S^1 \times R \to R \tag{5.28}$$

$$B_1 \in [-B, B) \tag{5.29}$$

$B_1$ and $x_1$ have two rest points: the origin, which is an elliptic centre, and $\{B_1=-B\equiv B,\ x_1=0\}$, which is a saddle.

The non-degeneracy condition is satisfied in this case[21]. From KAM theory we know that, if , is sufficiently small, the perturbed (area preserving) Poincaré map has a set of closed curves, of positive Lebesgue measure, close to the original set, filled with dense irrational orbits[22]. Since, for $B_1 = \pm B$, $x_1 = 0$ only if

i.e $$\frac{x_1^2}{2} = k'' - 2 = 0 \tag{163}$$

for $F(B_1, x_1)=k''=2$, system 5.31 has a pair of homoclinic orbits[23]:

$$k'' = 2, \pm\sqrt{2(k''-2)} = 0 \tag{5.31}$$

$$\{B^\circ, x^\circ\} = \{\pm 2\arctan(\sinh(t)), \pm 2sech(t)\} \tag{5.32}$$

Assume that the total "degree of disequilibrium" of system 5.26 is greater than 2. Then

$$t = \cosh^{-1}(k'') = \frac{1}{8}(k''-2) \tag{5.33}$$

is a positive constant. The results of the foregoing analysis can then be summarized as follows:

- equation 5.26 is a two degree of freedom Hamiltonian system with three conserved quantities;
- F has a pair of homoclinic orbits at the energy level $k''=2$;

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[21] See appendix 5.A.

[22] See Guckenheimer and Holmes, p. 219.

[23] See Guckenheimer and Holmes, p. 201.

- $G'(I)=8_2>0$, and
- for $k' > k''$, $l''$ is a constant.

Let

$$\{F, k^1\}(t+N^\circ) \tag{5.34}$$

denote the symplectic form

and define

$$\{F(B^\circ,x^\circ),k^1(B^\circ,x^\circ,(8_2 t+N^\circ),l'')\} = \frac{*F}{*B}\frac{*k^1}{*x} - \frac{*F}{*x}\frac{*k^1}{*B} \tag{5.35}$$

$$9(N^\circ)=\int_{-\infty}^{\infty}\{F,k^1\}(t+N^\circ)\,dt \tag{5.36}$$

Then[24], if Melnikov's function $9(N^\circ)$ has a simple zero (and is independent of $,$, for $,$ sufficiently small, system 5.26 has transverse homoclinic orbits on every energy surface $k' > k''$. By Smale-Birkhoff homoclinic theorem[25] this implies that the associated Hamiltonian vector field has Smale horseshoes at every "degree of disequilibrium" greater than two. Using 5.32 and 5.7, Melnikov's function 5.36 becomes equal to[26]

$$9(N^\circ)=\int_{-\infty}^{\infty}+\sin(2\arctan(\sinh(t)))''$$

where $l''$, $p'_1$ and $p'_2$ are given by equations 5.9; $B_1 = B^\circ$, $x_1 = x^\circ$ and

$$\cdot\left\{\frac{al'}{p'_1}+\frac{cl'}{p'_2}+\frac{bl'}{p'_1 p'_2}-\frac{l^*(a+b)}{p'_1}-\frac{l^*(c+b)}{p'_2}\right\}\pm \tag{5.37}$$

$$B_1 = \frac{1}{8_2}\left\{\frac{2(k'-2)}{p'_1 p'_2}\sin(8_2 t+N^\circ)\right\} = T\sin(8_2 t+N^\circ) \tag{5.38}$$

Melnikov's function can then be evaluated by the method of

$$9(N^\circ) = \frac{8_2}{l''\sqrt{(a+b)}}\left\{\left(a+\frac{b}{p'_2}\right)\left[\frac{8_2 b}{p'_1}+\frac{8_2 b}{p'_2}\right]\right\}^{1/2}\left[l^*(a+b)-l'\right]^{1/2}\left\{a+\right. \tag{5.39}$$

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[24] See Guckenheimer and Holmes, p.252.

[25] Guckenheimer and Holmes, p. 252.

[26] See appendix 5.B.

residues[27] to yield

$$\theta(N^\circ) = \pm 4a\left(l_1^* + 2\sqrt{l_1 l_2}\cos N^\circ\right)\cdot \left\{\frac{(1+T\sin N^\circ)^2 + (1+b_{21}T\sin N^\circ)^2}{(1+T\sin N^\circ)^2\left(1+b_{21}^* + 2\sqrt{b_{21}}\cos N^\circ\right)}\right\} \tag{5.40}$$

Since 5.40 has a simple zero for

$$N^\circ = \arccos\left\{-\frac{1}{T}\left(\frac{l_1 + l_2}{2\sqrt{l_1 l_2}}\right)\right\} \tag{5.41}$$

the perturbed Hamiltonian system has Smale horseshoes at every "degree of disequilibrium" greater than two, for , sufficiently small.

[27] See appendix 5.C.

## Appendix 5.A

The unperturbed Hamiltonian can be written as

$$F(B, x_1) + \omega_2 I = k^\circ \qquad (5.A.1)$$

and its vector field is

$$\dot{B} = 0 \qquad (5.A.2)$$

$$\dot{x}_1 = \frac{\partial B_1}{\partial N} = \omega_2 x_1(N) \qquad (5.A.3)$$

From 5.28 and 5.29 we know that $B_1$ is a periodic function and that $B_1 \in [-B,B)$. If we take the restriction $E_{k^\circ}^{N^\circ} = \{(B_1, x_1, N, I) \in S^2 \times \mathbb{R}^2 \mid I = I^\circ,\ N = N^\circ \in [0, 2\pi]\}$ and linearize the system 5.A.3 at $(B_1 = 0, x_1 = 0)$, we obtain

$$\dot{x}_1 = \frac{\partial B_1}{\partial N} = \frac{1}{\omega_2} \sin B_1(N)$$

The roots of 5.A.4 are

$$\begin{pmatrix} \dot{B} \\ \dot{\tilde{k}} \end{pmatrix} = \begin{pmatrix} 0 & \frac{-1}{\omega_2} \\ \frac{1}{\omega_2} & 0 \end{pmatrix} \begin{pmatrix} B \\ \tilde{k} \end{pmatrix} \qquad (5.A.4)$$

$$\mu_{1,2} = \pm i \frac{1}{\omega_2} \qquad (5.A.5)$$

i.e., they are imaginary with 0 real part. Hence (0,0) is an elliptic centre, sorrounded by closed orbits filled with dense orbits if $1/\omega_2$ is irrational. The linear Poincaré map is

$$P_{k^\circ} = \begin{pmatrix} \cos 2\pi\frac{1}{\omega_2} & \omega_2 \sin 2\pi\frac{1}{\omega_2} \\ -\frac{1}{\omega_2}\sin 2\pi\frac{1}{\omega_2} & \cos 2\pi\frac{1}{\omega_2} \end{pmatrix} \qquad (5.A.6)$$

If 5.A.3 is linearized at $(B_1 = B = B, x = 0)$, we obtain

$$\begin{pmatrix} \dot{B} \\ \dot{\tilde{x}} \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{\omega_2} \\ \omega_1 & 0 \end{pmatrix} \begin{pmatrix} B \\ \tilde{x} \end{pmatrix} \qquad (5.A.5)$$

and the roots are now real and distinct

$$\mu_{1,2} = \pm \frac{\omega_1}{\omega_2} \qquad (5.A.6)$$

Hence (-B,0) is a saddle. Since $B_1$ is periodic with period $2B$, the non-degeneracy condition of the unperturbed Poincaré map can be checked as follows: set $B_1 = 0$ $x_1 = \bar{x}_1 > 0$; we have

hence $$F(B_1, x_1 | B_1 = 0 \; x_1 = \bar{x}_1) = \frac{\bar{x}_1^2}{2} = k'' > 0 \qquad (5.A.7)$$

and, in general $$\bar{x}_1 = \sqrt{2\,k''} \qquad (5.A.8)$$

Therefore we set $$\left( \frac{x_1^2}{2} + 1 - \cos B_1 \right) = \frac{\bar{x}_1^2}{2} \qquad (5.A.9)$$

and $$J = \bar{x}_1 \qquad (5.A.10)$$

$F(B_1, x_1)$ is now equal to $x_1 = \sqrt{J^2 - 2 + 2\cos B_1}$, hence (5.A.11)

and $$\begin{matrix} J = 0 \\ B_1 = 0 \end{matrix} \quad \begin{matrix} = \\ = \end{matrix} \quad \begin{matrix} 0 \\ \gamma(J) = J \end{matrix} \qquad (5.A.12)$$

$$\gamma'(J) = 1 > 0 \qquad (5.A.13)$$

## Appendix 5.B

Melnikov's function is

$$9(\mathrm{N}^\circ) = \int_{-\infty}^{\infty} \{F, k^1\}\ (8_2 t + \mathrm{N}^\circ)\, dt \tag{5.B.1}$$

On the homoclinic orbit:

$$\text{and } \frac{*F}{*\mathrm{B}_1} = \sin \mathrm{B}_1 = \sin\ (\pm\ 2 \arctan\ (\sinh\ (t)) = \tag{5.B.2}$$

$$\pm \sin\ (2 \arctan\ (\sinh\ (t)))$$

$$\text{We also have } \frac{*F}{*x_1} = x_2 = \pm\ 2\ sech\ (t) \tag{5.B.3}$$

$$\frac{*k^1}{*x_1} = \frac{*k^1}{*l'} \cdot \frac{*l'}{*x_1} = \tag{5.B.4}$$

$$= "_{l1}\left[\frac{l'}{p'^2_1}a + \frac{l'}{p'^2_2}c + \frac{l'}{p'_1 p'_2}b - \frac{l^*(a+b)}{p'_1} - \frac{l^*(c+b)}{p'_2}\right]$$

and

$$\text{Thus } \frac{*k^1}{*\mathrm{B}_1} = \left\{\frac{*k^1}{*p'_1}\ \frac{*p'_1}{*\mathrm{B}_1} + \frac{*k^1}{*p'_2}\ \frac{*p'_2}{*\mathrm{B}_1}\right\} = \tag{5.B.5}$$

$$9(\mathrm{N}^\circ) = \int_{-\infty}^{\infty} (\pm \sin(2 \arctan(\sinh(t)))'\left[\frac{l'}{a^1} \cdot \frac{b}{p'^2_2}\right] + \frac{b_{12}}{p'^2_2}\left[l^*(a+b) - l'\left(a + \frac{b}{p'_1}\right)\right] \tag{5.B.6}$$

$$a^1 + \frac{l'}{p'^2_2}c + \frac{l'}{p'_1 p'_2}b - \frac{l^*(a+b)}{p'_1} - \frac{l^*(c+b)}{p'_2}\Bigg] \pm 2sec$$

$$(a+b) - l'\left(a + \frac{b}{p'_2}\right)\Bigg] + b_{12}\frac{l'}{p'^2_2} \cdot \left[l^*(a+b) - l'\left(a +\right.\right.$$

## Appendix 5.C

The evaluation of 9(N°) by the method of residues is rather laborious. In this appendix I will therefore state only the most relevant aspects of the procedure. The reader may refer to Smirnov, vol. 3.II, and to Dieudonné, chapter IX, for a detailed exposition of the theory of residues and of the theorems mentioned in this appendix.

9(N°) is a linear combination of six basis integrals:

$$\sin(2\arctan(\sinh(t)))\,\frac{l'}{p'_1} \qquad (5.C.1)$$

$$\sin 2\arctan(\sinh(t))\,\frac{p'_2}{p'_1} \qquad (5.C.2)$$

$$\sin(2\arctan(\sinh(t)))\,\frac{p'_2}{p'_1} \qquad (5.C.3)$$

$$sech(t)\,\frac{l'^2}{p'^2_1} \qquad (5.C.4)$$

$$sech(t)\,\frac{p'^2_2}{p'^2_1} \qquad (5.C.5)$$

$$sech(t)\,\frac{l'^2}{p'^2_1 p'^2_2} \qquad (5.C.6)$$

In expressions 5.C.1, 5.C.3, 5.C.4 and 5.C.5 $p'_1$ can be replaced by $p'_2$ without changing the integral. On the homoclinic orbit, total employment and prices are given by

$$l' = [l^* + \varepsilon_{l1}\, 2\, sech(t) + \varepsilon_{l2}\, T \cdot \cos(\theta_2 t + N^\circ)] \qquad (5.C.7)$$

$$p'_1 = [1 + 2\arctan(\sinh(t)) + b_{21}\, T \sin(\theta_2 t) + N^\circ) \qquad (5.C.8)$$

$$= [1 + 2b_{21}\arctan(\sinh(t)) + T \sin(\theta_2 t + N^\circ) \qquad (5.C.9)$$

Replace the real variable t with the complex variable s, defined as follows

$$s = te^{in} \quad t\,, R_+ \quad 0 \le n \le B \qquad (5.C.10)$$

thus

and setting $\frac{ds}{dt} = e^{in}$ ; $dt = ds\, e^{-in}$ (5.C.11)

we have $g = e^{in} = \cos n + i \sin n$ (5.C.12)

$$dt = ds\, g^{-1} \quad (5.C.13)$$

We can also set $s = gt$ (5.C.14)

hence $n_t = n_t + 2kB$ ; $0 \le n_t \le 2B$ ; $k = 0, \pm 1, \pm 2, ..$ (5.C.15)

$$S = (n_t + 2kB)g \quad (5.C.16)$$

Now, let us analyse the behaviour of the hyperbolic and circular functions of s in expressions 5.C.1-5.C.9. We have

$$sech\,(s) = \frac{2\,e^{s}}{e^{2s}+1} = \cosh^{-1}(s) = \frac{2\,e^{gn_t}+e^{g2kB}}{e^{2gn_t}e^{g4kB}+1} \quad (5.C.17)$$

for $n \neq (B/2), , \neq i$:

$$\lim_{k\to\pm\infty} \cosh(s) = 0$$

and for $n = (B/2), \ , = i$:

$$\lim_{n_t\to\frac{B}{2}} \cosh^{-1}(s) = \lim_{n_t\to\frac{B}{2}} \cos^{-1}(n_t) = \frac{\pm 2i}{(\pm i)^2+1} = \pm\infty \quad (5.C.19)$$

for $n \neq (B/2), \sinh(s) = \frac{e^{s}-e^{-s}}{2}$ (5.C.20)

$$\lim_{n_t\to\frac{3B}{2}}$$

For k=0 and $n_t=0$, sinh(0)=0. Hence sinh(s) is a monotonic increasing function of s. For $\lim_{k\to\infty} \sinh(s)=\infty$ $\lim_{k\to-\infty} \sinh(s)=-\infty$ n=(B/2), ,=i: (5.C.21)

$$\sinh(s)=\frac{e^{2i(n_t+2kB)}-1}{2e^{i(n_t+2kB)}}=i\sin(n_t)=-\frac{1}{i}\sin(n_t) \quad (5.C.21)$$

and

$$\lim_{n_t\to\frac{B}{2}} i\sin(n_t)=i \quad (5.C.22)$$

$$\lim_{n_t\to\frac{3B}{2}} i\sin(n_t)=-i \quad (5.C.23)$$

Set

$$2\arctan(\sinh(s))=2R \quad (5.C.24)$$

For $n\neq(B/2), ,\neq1$, we have

$$2R=\frac{1}{i}\ln\frac{i-\sinh(gt)}{i+\sinh(gt)} \quad (5.C.25)$$

Since

$$\left|\frac{1-\sinh(gt)}{1+\sinh(gt)}\right|=\frac{\sqrt{-1+\sinh^2(gt)}}{\sqrt{-1+\sinh^2(gt)}}=1 \quad (5.C.26)$$

we have

where $$2R=\frac{1}{i}arg\left(\ln\frac{i-\sinh(gt)}{i+\sinh(gt)}\right)=n_R+2k_RB \quad (5.C.27)$$

$$0\leq n_R\leq 2B;\quad k_R=0,\pm1,\pm2\ldots \quad (5.C.28)$$

Setting $k_R=0$, R becomes a real one-one function of s. For n = (B/2), ,=i

is a complex one-one function and

$$2R=\frac{1}{i}\ln\frac{i-i\sin(n_t)}{i+i\sin(n_t)}=\frac{1}{i}\ln\frac{1-\sin n_t}{1+\sin n_t} \quad (5.C.29)$$

hence $$\lim_{n_t \to \frac{B}{2}} \frac{1}{i} \ln \frac{1-\sin n_t}{1+\sin n_t} = -\infty \quad (5.C.30)$$

$$\lim_{n_t \to \frac{3B}{2}} 2 \frac{1}{i} \arctan(i) \frac{1-\sin n_t}{1+\sin n_t} = -\infty \; +\infty \quad (5.C.31)$$

$$2 \arctan(-i) = \infty \quad (5.C.32)$$

Thus

$$\sin(2 \arctan(\sinh(s))) = \sin(2R) \quad (5.C.33)$$

for $n \neq (B/2), , \neq i$

$$\sin(2R) = \sin(2n_R) \quad (5.C.34)$$

is a real periodic function. For $n = (B/2), , = i$

thus $$\sin(R)\,\sin\left(-i\ln\frac{1-\sin N_t}{1+\sin N_t}\right)=$$

$$=\frac{\left(\frac{1-\sin N_t}{1+\sin N_t}\right)^{i}-\left(\frac{1+\sin N_t}{1-\sin N_t}\right)^{i}}{2i}=\frac{e^{i\ln\frac{1-\sin N_t}{1+\sin N_t}}-e^{-i\ln\frac{1-\sin N_t}{1+\sin N_t}}}{2i}=\frac{\left(\frac{1-\sin N_t}{1+\sin N_t}\right)^{i}-\left(\frac{1+\sin N_t}{1-\sin N_t}\right)^{i}}{2i}=\frac{1-\sin^2 n_t}{\cos^2 n_t}$$

which implies that

$$\sin(2R)=\frac{\cos^2 n_t}{1-\sin^2 n_t}=\frac{1-\sin^2 n_t}{1-\sin^2 n_t}=1 \qquad (5.C.35)$$

$$\lim_{n_t\to 0}\sin(2R)=0 \qquad (5.C.36)$$

$$\lim_{n_t\to \frac{\pi}{2}}\sin(2R)=+\infty \qquad (5.C.37)$$

$$\lim_{n_t\to \frac{3\pi}{2}}\sin(2R)=-\infty \qquad (5.C.38)$$

In a similar way it can be proved that if

$$s=t\in\mathbb{R}_+ \qquad (5.C.39)$$

i.e., for

$$n=0\ ;\ n=\pi \qquad (5.C.40)$$

$\cos(\omega_2 s+N)$ and $\sin(\omega_2 s+N)$ are periodic functions with unit module. If

$$s=t(\cos n+i\sin n);\quad t\in\mathbb{R} \qquad (5.C.41)$$

the modules of $\cos(\omega_2 s+N)$ and $\sin(\omega_2 s+N)$ are increasing functions of $t\in\mathbb{R}_+$. Expression

can be set equal to the sum of $\sin(2\arctan(\sinh(t)))$ $\frac{l'}{p'^2_1}$ (5.C.42)

and $$\frac{2''_{l1}\sin 2R}{\cosh(s)[1+2R+b_{21}T\sin(\omega_2 s+N^{\circ})]^2} \qquad (5.C.43)$$

(5.C.44)

For $n=0$ or $n=B$ 5.C.44 becomes

$$\frac{\sin 2R[\quad + \quad T\cos(8_2 s + N^\circ)]}{[1+2R+b_{21}T\sin(8_2 s+N^\circ)]^2} \tag{5.C.45}$$

Since

$$\frac{2\quad \sin(n_R)}{\cosh(t)[1+n_R+b_{21}T(\sin 8_2 t\cos N^\circ+\sin N^\circ\cos 8_2 t)]^2} \tag{5.C.46}$$

5.C.46 tends uniformly to 0 for $t\to\infty$. For $0<n_t<B$, $n\neq(B/2)$, 5.C.44 becomes

$$\lim_{n_t\to\pm\infty}\cosh(t) = \infty \tag{5.C.47}$$

and the foregoing analysis of cosh(s) and $\sin(8_2 s+N^\circ)$ shows that 5.C.48 also tends uniformly to 0 for $t\to\infty$. For $n=(B/2)$, $\quad = i$ and $n_t\neq(B/2)$, $n_t\neq(3B/2)$ we have

$$\frac{2\quad \sin n_R}{\quad\cosh(t)[1+n_R+b_{21}T\sin(8_2 t+N^\circ)]^2} \tag{5.C.48}$$

Since

$$\frac{1}{i}\frac{\sin n_t}{\cos^2 n_t}\;\frac{2\quad}{\cos n_t\left[1+\frac{1}{i}e^{n_i}\frac{1-\sin n_t}{1+\sin n_t}+b_{21}T\sin(8\right.} \tag{5.C.49}$$

$$\lim_{t\to\infty}\sin(i8_2 t+N^\circ)_t\left[\lim_{t\to\infty}ri\cos N^\circ\sinh t+\sin\right. \tag{5.C.50}$$

5.C.49 also tends uniformly to 0 for $t\to\infty$.

Set

$$F(s) = \frac{2\quad \sin 2R}{\cosh(s)\,[1+2R+b_{21}T\sin(8_2 s+N^\circ)]^2} \tag{5.C.51}$$

then $\lim_{s\to\infty} sF(s) = 0$, because

$$\lim_{s\to\infty}\frac{1}{s}=0 \tag{5.C.52}$$

for any $n>0$. Moreover, since we have set $k_Q=0$, $F(s)$ is analytic everywhere in the upper half-plane, except at the poles

$$s = i\left(\frac{B}{2}+2kB\right) \tag{5.C.53}$$

and

$$s = i\left(\frac{3B}{2}+2kB\right) \tag{5.C.54}$$

The two poles of $F(s)$ lie in the upper half-plane are uniquely determined once k is determined. The corresponding values of the residues of $F(s)$ are invariant for all values of $k\in\mathbb{I}_+$; thus, to evaluate the residues, k can safely be set equal to 0. From the

theory of residues[28] we know that

$$\lim_{r\to\infty}\int_{-r}^{r} F(t)dt = 2Bi\sum a_i \qquad (5.C.55)$$

where Ea is the sum of the residues of F(s) at the poles in the upper half-plane. The residues can be evaluated as the limits for

$n_t \to B/2$ and $n_t \to 3B/2$ of

$$\frac{1}{2}\frac{d^{n-1}}{dn_t^{n-1}}\left\{\frac{1}{i}\frac{2\sin n_t}{\cos^3 n_t}\frac{(n_t - n_t^0)^n}{[1+\frac{1}{i}\ln\frac{1-\sin n_t}{1+\sin n_t}+b_{21}T\sin(\quad)]^n}\right\}_{\substack{n=1,2\\ j=1,2}} \qquad (5.C.56)$$

where $n^1_t$ = B/2 and $n^2_t$ = 3B/2. Since these limits are respectively equal to $+\infty$ and $-\infty$, their sum is equal to 0. Hence, the principal value of the integral of 5.C.44 is 0.

In a similar way it can be proved that the principal value of the integrals of 5.C.45, 5.C.2, 5.C.3, 5.C.4 and 5.C.6 are all equal to 0. In order to evaluate the integral of expression 5.C.5 by the method of residues, let us again substitute the complex variable s for the real variable t. We obtain

In this case also:

$$F(s) = \frac{[l^*+2\text{"}_{l1}\cosh^{-1}(s)+\text{"}_{l2}T\cos(8_2 s+N^\circ)]^2}{\cosh(s)[1+2R+b_{21}T\sin(8_2 s+N^\circ)]^2} \qquad (5.C.57)$$

$$\lim_{s\to\infty} s\,F(s) = 0 \qquad (5.C.58)$$

everywhere in the upper half-plane except at n=B/2, where 5.C.57 is equal to

Let us consider the following function:

$$\frac{\{l^*+2\text{"}_{l1}\cos^{-1}n_t+\text{"}_{l2}T\cos[i8_2(n_t+2kB)+N^\circ]\}^2}{\cos n_t\{[1+\frac{1}{i}\ln\frac{1-\sin n_t}{1+\sin n_t}+b_{21}T\sin[i8_2(n_t+2kB)+N^\circ} \qquad (5.C.59)$$

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[28] See Smirnov, 1982, 3.II, p. 227.

$F_1(s)$, is continuous and has continuous first derivatives for all values of $n \in (0,B)$, except at the poles; moreover

$$-F\left(\frac{B}{2}, t \mid t \in \mathbb{R}_+\right) + F\left(n, t \mid \frac{B}{2} < n \le B; t \in \mathbb{R}_+\right) \tag{5.C.60}$$

$$\lim_{s \to \infty} s F_1(s) = 0 \tag{5.C.61}$$

uniformly in the same domain. Hence[29]

$$\lim_{r \to \infty} \int_{C_r} F_1(s)\, ds = 0 \tag{5.C.62}$$

where $C_r$ is the semicircle determined by $0 \le n \le B$ and $r = |s|$. On the real axis $F_1(s)$ may be written as

Since the residues of $F\left(n, t \mid t \in \mathbb{R}_+\right)$ in $F\left(\frac{B}{2}, t\right)$ in the upper half-plane cancel out, we must have

$$+ F(B, t \mid t \in \mathbb{R}_+) = F(t \mid t \in \mathbb{R}) - F\left(\frac{B}{2}, 0\right) \tag{5.C.63}$$

Hence

$$\lim_{r \to \infty} \left\{ \int_{-r}^{r} F(t \mid t \in \mathbb{R})\, dt - F\left(\frac{B}{2}, 0\right) + \int_{C_r} F_1(s)\, ds \right\} = 0 \tag{5.C.64}$$

$$\lim_{r \to \infty} \int_{-r}^{r} F(t)\, dt - F\left(\frac{B}{2}, 0\right) = 0 \tag{5.C.65}$$

and therefore

$$\lim_{r \to \infty} \int_{-r}^{r} F(t)\, dt = F\left(\frac{B}{2}, 0\right) = \frac{[l^* + 2\pi\nu_{l2} - \pi\nu_{l2} T \cos N^\circ]}{[1 + b_{21} T \sin N^\circ]^2} \tag{5.C.66}$$

Thus, Melnikov's function is reduced to

which has a simple zero for

$$g(N^\circ) = \pm 4\pi \left\{ \frac{[l^* + 2\pi\nu_{l2} - \pi\nu_{l2} T \cos N^\circ]^2}{[1 + b_{21} T \sin N^\circ]^2} \right\} + \tag{5.C.67}$$

$$+ N^\circ = b_{12} \frac{[l^* + 2\pi\nu_{l2} - \pi\nu_{l2} T \cos N^\circ]^2}{[1 + T \sin N^\circ]^2} \arccos\left( \frac{l^* + 2\pi\nu_{l2}}{\pi\nu_{l1} T} \right) \tag{5.C.68}$$

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[29] See Smirnov, 1989, 3.II, pp.223-229.

## 6. **Conclusions**.

"Why is economics like the weather? Because both are highly irregular if not chaotic, thus making prediction unreliable or even impossible."[30] This statement is probably an effective synthesis of Goodwin's views on the dynamics of modern economies. These views, or perhaps this "philosophy", appear to the interested scholar like the leading thread of Goodwin's lifelong work on dynamics.

In this paper it has been shown that this "philosophy" is not incompatible with the usual economic axioms, which state that agents aim at maximizing either profits, or utility, or both. It has also been shown that symplectic transformations of coordinates are an essential tool of analysis when a Hamiltonian adjustment process is assumed, as is the case of Goodwin's 1953 "Walrasian" model. Finally, it has been shown that, if real wages are not entirely flexible and returns to scale are not constant, the adjusment process can generate very complex, and in extreme cases chaotic, motions.

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[30] Goodwin, 1990, p. 1.

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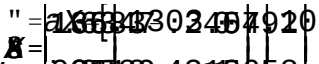
TABLE 1

**Parameters of the two-sector model.**



**Initial Conditions**

| Variable | Models | | |
|---|---|---|---|
| | Navajo | Sioux | Paiute |
| B | .12 | -.1 | .12 |
| B | -.16 | .1 | -.16 |
| x | -.15 | - | -.15 |
| x | .09 | - | .09 |
| l' | - | .4402 | .4402 |
| l' | - | -.001 | - |
| B | - | .02 | - |
| B | - | -.03 | - |

**Simultaneous Adjustment of Quantities and Prices: an Example of Hamiltonian Dynamics.**

Paola Antonello

Abstract

In a well-known 1953 essay, "Static and Dynamic Linear General Equilibrium Models", Richard Goodwin analyses the dynamic adjustment of quantities and prices to long period equilibrium, in a set of n "Walrasian" markets. He treats the crossed adjustment of prices and quantities as a linear Hamiltonian vector field. In more recent works Goodwin has introduced non-linear perturbations in his multisectoral adjustment models, by assuming that real consumption depends non-linearly on relative prices. Goodwin's use of Hamiltonian dynamics and of symplectic coordinates changes opens up a wide range of fascinating potential developments for the analysis of adjustment processes in multisectoral systems subject to real perturbations. It has, however, not been totally exempted from objections, usually referring to the lack of microfoundations of his macro dynamic analysis and to his use of non-Cartesian coordinate systems in economics.

The aim of this paper is threefold: i) to investigate whether Goodwin's behavioural hypotheses are compatible with the assumption that agents maximize. ii) To show that, if the dynamic process is Hamiltonian, symplectic coordinates changes are essential tools of analysis. iii) To analyse the dynamic behaviour of Goodwin's Hamiltonian model, subject to the non-linear perturbation he suggests in Chaotic Economic Dynamics (1990), and to point out some of the developments this approach may lead to.

Goodwin's 1953 cross-dual model is at first derived from an optimal control model, the objective functional of which is the aggregate excess profits function, i.e. the sum of sectoral excess profits. This model generates simple harmonic motions. In the second part of the paper, the following assumptions are introduced: a) the nominal wage is fixed and entirely spent for consumption; b) consumer utility functions are Cobb-Douglas; c) consumers aim to maximize their current utility. Under these hypotheses, Goodwin's 1990 consumer demand functions are easily derived. Since prices and the real wage are flexible, the closed orbits solutions disappear and long period equilibrium becomes asymptotically stable.

In the last section of the paper, the analysis is restricted to a two sector model. Goodwin's basis hypotheses are slightly modified. It is assumed that the unperturbed two degree of freedom Hamiltonian is non-linear and has a homoclinic orbit. It is further assumed that the perturbation is itself periodic. The economic meaning of these assumptions is that returns to scale are not constant and that real wages are partially rigid. By applying Melnikov's method, it is then proved that the perturbed Hamiltonian system has transverse homoclinic orbits and, therefore, Smale

horseshoes. Hence, under the assumptions of variable returns to scale and of real wage rigidity the model can generate chaotic transients or pure chaotic motions.

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