A feedback control approach to multistage production planning based on input-output analysis

Jean-Claude Hennet

LAAS-CNRS, 7, Ave. du Colonel Roche, 31077 Toulouse Cédex 4, FRANCE
e-mail : hennet@laas.fr

Abstract

A discrete-time linear model of a manufacturing facility is constructed from an input-output analysis of the system. The model represents the proportionality dependency between the inputs and the outputs, and the lead-times between order release and order completion. An extended state representation of the model is constructed, and a state feedback control approach is proposed.

Using positive invariance relations for polyhedral domains, conditions are provided under which the respect of production and inventory capacity constraints are satisfied by the system trajectory. Robustness with respect to demand fluctuations and lead-time variations is characterized by levels of admissible bounds on persistent disturbance inputs.

A solution to the constrained control problem is proposed and positive invariance domains associated to this control are constructed.

Keywords
Input-Output Analysis, Multistage Production Systems, Production Planning, Delay Systems, State feedback, Positive Invariance

1 Introduction

A convenient technique for integrating the timing aspects in the input-output analysis of a multi-level production system is to represent the system by a MIMO (Multi Input - Multi Output) linear dynamical model with delays. To adequately represent a production-inventory system, such a model should also include constraints on stock levels and production capacity.

In general, the purpose of a production plan is to combine production and inventory management so as to find a good trade-off between maximizing the probability to satisfy the demand without delay and minimizing the inventory costs. Due to its complexity, the planning problem is usually formulated over different time horizons and for different levels of product aggregation. The most classical scheme consists of solving an aggregate planning problem and deriving from it a detailed production plan.

This classical scheme can be improved by directly using the final demand in the detailed planning problem and solving a single optimization problem. This scheme can be achieved in a systematic way using the discrete-time dynamic model presented in section 2. Such a model could be used in a classical open-loop predictive framework. An optimal detailed plan would then be computed on the basis of forecasts of future external demand, productivity and lead-times. Some major drawbacks of this approach, in a real environment, are the need for frequent replanning, and the risk of nervousness of the production policy. These drawbacks are strongly connected to the "open-loop" feature of classical planning techniques.

A natural consequence of this analysis is to try to apply feedback control techniques to production planning problems. In reality, some system feedback always has to be introduced to provide the production system with reactivity. The MRP technique itself tends to implement feedback on stock levels, but is is only partly reactive, because it combines such closed-loop measurements with open-loop data on forecasted demands and requirements.

The Kanban technique can be seen as a true feedback control technique, based both on real stocks and real demands measurements. It is reactive and robust, but limited to manufacturing systems with a simple structure and a regular demand pattern. Rather than a reactive planning technique, it can be seen as an attempt to suppress the planning stage, sometimes with the drawback of shortsightedness.

The approach described in section 3 relies on the construction of a feedback control policy based on a state formulation of the MIMO model of the multi-level manufacturing system. The feedback data used by the control policy are past production decisions and observation of current stock levels. The respect of constraints on inventory and production capacity under limited fluctuations of the external demand is guaranteed by the satisfaction of linear positive invariance conditions associated to a polyhedral domain of admissible states of the system. The main advantage of the invariance approach for linear systems and polyhedral regions, is that the whole control problem can be formulated as a set of linear relations and solved by Linear Programming. Examples of such domains and associated feedback gain matrices defining the production strategy are finally presented.

Notations. By convention, inequalities between vectors and inequalities between matrices are componentwise. In particular, a (componentwise) nonnegative matrix is a matrix having all its components nonnegative. The absolute value $|v|$ of vector v is defined as the vector of the absolute value of its components. Matrix I denotes the identity matrix of appropriate dimension; matrix $\mathcal{O}$ denotes the zero matrix of appropriate dimension; vector $\mathbf{1}$ is the unity vector of appropriate dimension, defined by $\mathbf{1} = \begin{bmatrix} 1 & \ldots & 1 \end{bmatrix}^T$. The advance operator, q, applies to vector sequences (v_k) under the form: $qv_k = v_{k+1}$. The delay operator is the inverse operator, denoted q^{-1}, such that: $q^{-1}v_k = v_{k-1}$. The ℓ^∞ norm is classically defined for bounded vector sequences $h = \{h_k\}$, with $h_k = [h_{1k} \quad h_{pk}]$ by: $\|h\|_{\ell^\infty} = \sup_{k \in \mathcal{N}} \|h_k\|_\infty < \infty$, under the classical definition of the infinity norm : $\|h_k\|_\infty = \max_{i=1,\ldots,p} |h_{ik}|$.

2 The multistage production model

The use of input-output analysis to construct models of production and inventory systems has become classical since the work of Veinott [Vei69]. Then, in the late 1980's, Grubbström and his co-authors [GO92, GM94] have introduced lead-times in their input-output based models, under the form of constant delays. In the framework of continuous-time systems with delays, they have made use of the Laplace transform to analyze several production policies and to provide economic interpretations and evaluations [GM94, Mol95]. In this paper, the discrete-time framework is preferred in relation with feedback control objectives, because it allows for a finite dimensional state space description of the system, under the assumption of commensurate delays.

2.1 The static Input-Output model

A *Bill of Materials* can be constructed to summarize the input and output flows of a manufacturing system over some working horizon, $\mathcal{T}$. Let z denote the cumulated external output vector and u the cumulated production/supply vector over $\mathcal{T}$. Then, a static Leontief-type representation of the system over $\mathcal{T}$ is :

$$(\mathcal{R} - \mathcal{S})u = z \tag{1}$$

where $\mathcal{R}$ and $\mathcal{S}$ are nonnegative matrices. $\mathcal{R}$ is the output matrix (for internal and external transformations) and $\mathcal{S}$ the (internal) consumption matrix. The *net production* matrix, $(\mathcal{R} - \mathcal{S})$ relates u to z. For general manufacturing systems, matrices $\mathcal{R}$ and $\mathcal{S}$ have the same rectangular structure. If matrix $(\mathcal{R} - \mathcal{S})$ is not full row-rank, it represents the possibility to obtain the same combination of outputs through different activities. If $(\mathcal{R} - \mathcal{S})$ is not full column-rank, it corresponds to the possibility to substitute some production inputs by some other ones.

The product structures studied in this paper are generalized assembly graph. Each production activity has several input products, and products of the same type may be used by several activities. However there is a one-to-one correspondence between output products and activities: each production activity has exactly one output product. According to a given manufacturing recipe, the production of one unit of product i consumes components $j = 1, ..., N$ in quantities π_{ji}, for $j = 1, ..., N$. Such production structures can be represented by *Gozinto graphs* [Vaz55], which typically characterize the field of application of Requirements planning, and particularly MRP techniques (see e.g. [Bak93]). They characterize what is sometimes called the *explosive industries* [Bur89], because each intermediate or final product can be *exploded* into its components. The input and output matrices representing such generalized assembly graphs are square matrices. If activities and associated output products have the same index, matrix $\mathcal{R}$ is the identity matrix, I, and matrix $\mathcal{S}$ is matrix $\Pi = ((\pi_{ji}))$.

It is also assumed that the graph of the product structure has no circuit. Then its places can be partitionned by levels. Level 0 is for final products, level l for products which are components of products of levels stricly less than l and of at least one level $l - 1$ product. Products being ordered in agreement with the decreasing order of their level, the input matrix Π is nonnegative, upper triangular with zeros on the main diagonal:

$$\Pi = \begin{bmatrix} 0 & \pi_{12} & \dots & \pi_{1N} \\ \vdots & \ddots & \ddots & \vdots \\ & & 0 & \pi_{N-1,N} \\ 0 & \dots & & 0 \end{bmatrix}. \tag{2}$$

The net production matrix, $(I - \Pi)$ has all its off-diagonal elements nonpositive. It may be called an *essentially nonpositive* matrix [BP79]. Furthermore, it is upper-triangular, and its diagonal elements are equal to 1. Therefore, $\det(I - \Pi) = 1$, and more generally, all its principal minors are equal to 1. It is a well-known characteristic property of an M-matrix (also called a class K matrix [KWW75]) to be an essentially nonpositive matrix with all its principal minors positive. Therefore, matrix $(I - \Pi)$ is an M-matrix. This implies, in particular, that its inverse $(I - \Pi)^{-1}$ is a nonnegative matrix [BP79].

This property can be easily checked directly :

$$(I-\Pi)^{-1} = \begin{bmatrix} 1 & b_{12} & \dots & b_{1N} \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & b_{N-1,N} \\ 0 & \dots & 0 & 1 \end{bmatrix}.$$

The following relations are obtained :

$$\begin{cases} b_{i,i+1} & = & \pi_{i,i+1} \text{ for } i = 1, \dots N-1 \\ b_{i,i+k} & = & \pi_{i,i+k} + \sum_{j=1,\dots,k-1} \pi_{i,i+j} b_{i+j,i+k} \text{ for } i = 1, \dots N-2, \ k = 2, \dots, N-i. \end{cases}$$

From these relations, it is easy to show by induction that since matrix Π is nonnegative, matrix $(I-\Pi)^{-1}$ is also nonnegative.

An other characteristic property of an essentially nonpositive matrix, M, which is an M-matrix is the existence of a nonnegative vector $x \neq 0$ such that $Mx > 0$. This property, together with the fact that each column of $(I-\Pi)$ has exactly one positive element, shows that $(I-\Pi)$ is a Leontief matrix [KWW75].

2.2 A dynamic Input-Output representation

The evolution of production and stocks is described by a discrete-time model. In this model, all the production and transportation delays, as well as set-up times, are supposed to be known, constant and multiple of the elementary time unit. Such an assumption of constant lead-times, independent of production levels is classical in requirements planning. In reality, as demonstrated in [Kar93], lead-times are made of two terms : manufacturing times, which may often be assumed constant and waiting times, which are strongly load-dependent, specially in heavy utilization conditions. This is one of the reasons why it is important to build a closed-loop policy which is sufficiently robust to parameter uncertainties and which has good disturbance attenuation properties.

Each time unit is indexed by k, with $k \in \mathcal{N}$. To each product i , $i = 1, ..., N$ is associated at time k

- an external demand, d_{ik}, such that, in general, $d_{ik} = 0 \ \forall k \in \mathcal{N}$ if i is not a final product,
- a production level, u_{ik}, generating the output u_{ik} at period $k + \theta_i$; by definition, θ_i is the production delay (or lead-time) for product i,
- the stock level at the end of period k, s_{ik}.

A possible representation of the system dynamics is based on timed-Petri nets. A Petri net representation of a multistage product structure is represented on Fig.1. From the structural viewpoint, such a Petri net is characterized by the one-to-one correspondence between each transition and its output place. Then, by convention, a transition and its output place carry the same number and a unit weight can be put on each transition-place arc. The components for which $\pi_{ji} \neq 0$ are associated to the input places of transition i. Each place which is not a sink may be an input place to several transitions. The correspondence between the input-output model and the Petri net model is then direct :

- the "Pre" matrix of the Petri net is the input matrix, Π,
- the "Post" matrix of the Petri net is the output matrix, I,
- the incidence matrix of the Petri net (Post - Pre) is the net production matrix, $(I - \Pi)$.

The controllability properties of Petri nets for which the outdegree of each transition is 0 or 1 have been studied in other contexts, particularly in models of logic programs representing Horn-clause predicates [Mer76].

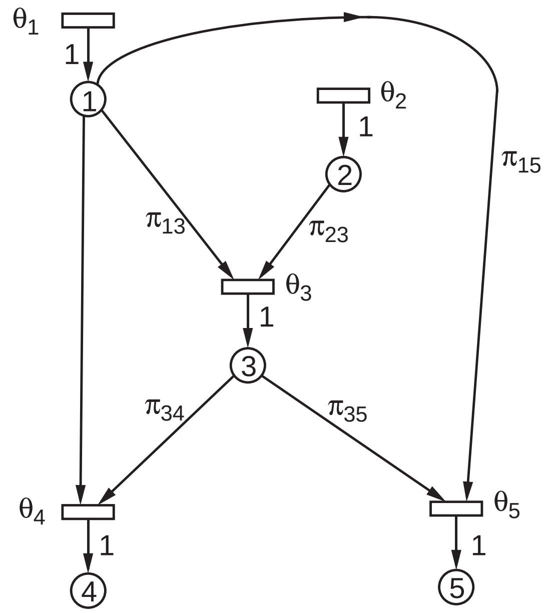

Figure 1: A Petri net representation of a product structure

In addition to a logical interpretation of the product structure, the Petri net representation provides a quantitative model of the production system, through the marking evolution equation. The marking variables considered here are real and nonnegative. They correspond to an extension of Petri nets known as *continuous Petri nets* [VH86], [HV87], [DA87]. This extension is combined with the use of timed-transitions : in order to produce the output u_{ik} at period $k+\theta_i$, the components $j = 1, ..., N$ are required in quantities $\pi_{ji}u_{ik}$ at time k. The stock level, s_{ik} corresponds to the marking of place i at period k. Adding additional instantaneous transitions for the external demand at each place, the marking evolution equation takes the form:

$$s_k = s_{k-1} + [diag(q^{-\theta_i}) - \Pi]u_k - d_k \tag{3}$$

From the Petri net representation of the system, it is clear that transition enabling conditions are equivalent to the conditions of nonnegativity of stock levels at each instant : $s_{ik} \geq 0 \; \forall i = 1, ..., N$, $\forall k \in \mathcal{N}$.

The existence of safety stocks is a classical technique to react to occasional demand fluctuations. Nominal stock levels are usually related to the probability distributions of final demands. They can be computed so as to achieve the best trade-off between expected inventory costs and expected stock-out costs over the planning horizon (see e.g. [FH83]).

A nominal safety stock level, $s_i^* \geq 0$ can be considered for each product. In order to take into account the possibility of backorders, a modified balance equation can be written, with the components y_{ik} of vector y_k not necessarily nonnegative :

$$y_k = y_{k-1} + [diag(q^{-\theta_i}) - \Pi]u_k - d_k \tag{4}$$

The actual stock level can be deduced from equation (4) as follows :

$$\begin{cases} s_{ik} &= y_{ik} + s_i^* & \text{if } y_{ik} \geq -s_i^* \\ s_{ik} &= 0 & \text{with a backorder of } -y_{ik} - s_i^* \text{ if } y_{ik} \leq -s_i^*. \end{cases} \tag{5}$$

Equation (4) can be re-written using the delay operator q^{-1}:

$$(1 - q^{-1})y_k = T(q^{-1})u_k - d_k \tag{6}$$

Matrix $T(q^{-1})$ is called the dynamic net production matrix. It is defined by:

$$T(q^{-1}) = diag(q^{-\theta_i}) - \Pi \tag{7}$$

In the sequel, matrix $T(q^{-1})$ is decomposed as follows:

$$T(q^{-1}) = T_0 + q^{-1}T_1 + ... + q^{-\theta}T_\theta \tag{8}$$

where $T_0 = -\Pi$, $(T_j) = diag(t_i^j)$ with $t_i^j = 1$ if and only if $\theta_i = j$, $t_i^j = 0$ in any other case, $\forall j = 1, ..., \theta$. The system delay, denoted θ, is defined as the maximal lead-time :

$$\theta = \max_{i=1}^{N} \theta_i. \tag{9}$$

2.3 The nominal activity levels

The production planning problem is now considered as a control problem, with y_k as the output vector, d_k as an external input vector and u_k as the control vector.

The demand vector is supposed random but stationary. It is decomposed as follows:

$$d_k = d - w_k \tag{10}$$

with mean values $\bar{d}_k = d$, $\bar{w}_k = \mathbf{0}$.

In addition to demand net fluctuations for final products, the disturbance vector w_k may include disturbances on supply and production of intermediate products, and may represent the effects of resource capacity variations and unpredicted delays. The disturbance vector w_k is supposed totally random and possibly time-varying. The only restrictive (but realistic) assumption is that w_k is bounded. At any period k, this vector may take any value in the polyhedral domain $R[L, \lambda]$, with $L \in \Re^{M \times N}$, $M \geq N$, rank$(L) = N$, λ a nonnegative vector in $\Re^M$:

$$w_k \in R[L, \lambda] = \{\omega \; ; \; L\omega \leq \lambda\} \text{ with } L = \begin{bmatrix} I \\ -I \end{bmatrix}, \; \lambda = \begin{bmatrix} \overline{\omega} \\ \underline{\omega} \end{bmatrix}. \tag{11}$$

Using the decomposition (10), the problem can be transformed into a regulation problem by introducing the translated control vector:

$$v_k = u_k - u^* \tag{12}$$

where u^* is the steady-state nominal control vector associated with the stationary mean demand vector d:

$$u^* = [diag(q^{-\theta_i}) - \Pi]^{-1} d = (I - \Pi)^{-1} d \tag{13}$$

Nonnegativity of $(I-\Pi)^{-1}$ is a direct consequence of the fact that $(I-\Pi)$ is an M-matrix. Thus, $d \geq 0$ implies from (13) $u^* \geq 0$.

Therefore, for a multistage production system described by input matrix Π of (2) and output matrix I, vector u^* is well-defined by relation (13) and nonnegative.

2.4 The state model

Using the change of variable (12), equation (6) is re-written in the form:

$$(1 - q^{-1}) y_k = T(q^{-1}) v_k + w_k \tag{14}$$

At time k, the past evolution of the system can be summarized by the state vector :

$$x_k = [y_{k-1} \ v_{k-1} \ \cdots v_{k-\theta}]^T \tag{15}$$

The system dynamics are described by the discrete-time state equation :

$$x_{k+1} = \mathcal{A} x_k + \mathcal{B}_1 w_k + \mathcal{B}_2 v_k \tag{16}$$

with

$$\mathcal{A} = \begin{bmatrix} I & T_1 & \cdots & \cdots & T_\theta \\ \mathcal{O} & \cdots & & \cdots & \mathcal{O} \\ \mathcal{O} & I & \mathcal{O} & \cdots & \mathcal{O} \\ \vdots & & \ddots & \ddots & \vdots \\ \mathcal{O} & \cdots & \mathcal{O} & I & \mathcal{O} \end{bmatrix}, \ \mathcal{B}_1 = \begin{bmatrix} I \\ \mathcal{O} \\ \vdots \\ \\ \mathcal{O} \end{bmatrix}, \ \mathcal{B}_2 = \begin{bmatrix} T_0 \\ I \\ \mathcal{O} \\ \vdots \\ \mathcal{O} \end{bmatrix}.$$

2.5 The domain of constraints

The physical meaning of stock and production variables imposes nonnegativity constraints. The expression of these constraints should be adapted to the definition of the variables in model (14).

- From the change of variables (12), nonnegativity of the production vector u_k can be achieved by imposing the constraint :

$$v_k \geq -u^* \tag{17}$$

- The possibility of having backorders is normally restricted to final products. For primary and intermediate products, the nonnegativity constraints on stocks is formulated as in (5):

$$y_{ik} \geq -s_i^* \quad \forall k \in \mathcal{N}. \tag{18}$$

Aggregated production capacity constraints are associated with the R resources, and similarly, stock capacity constraints are associated with the P storage zones. Variable u_{ik} represents the production release order for product i at time k. Production is actually running at periods

$k, ..., k+\theta_i-1$, and delivered at period $k+\theta_i$. Assuming a constant use of resources along the production cycle, resource capacity constraints take the following form, for $r=1,...,R$, $k \geq \theta$:

$$\sum_{i=1}^{N}\sum_{l=1}^{\theta_i} m_i^r u_{i,k-l} \leq M_k^r \tag{19}$$

m_i^r is the amount of ressource r currently needed to produce one unit of product i.
M_k^r is the capacity of resource i at period k.

If backorders are not allowed, storage capacity constraints take the form :

$$\sum_{i=1}^{N} n_i^p y_{ik} \leq N_k^p - \sum_{i=1}^{N} n_i^p s_i^* \tag{20}$$

n_i^p is the capacity at stock p required for one unit of product i. System (16) is subject to the set of linear constraints (17), (18), (19) and (20), written in the state form

$$x_k \in R[Q,\rho] = \{x; Qx \leq \rho\} \tag{21}$$

$$\text{with } Q = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}, \rho = \begin{bmatrix} \rho_1 \\ \rho_2 \end{bmatrix}, \quad dim(Q) = q \times n, \quad dim(\rho) = q$$

$$\text{and } Q_1 = \begin{bmatrix} N & \mathcal{O} & \cdots & \mathcal{O} \\ \mathcal{O} & M_1 & \cdots & M_\theta \\ \vdots & \ddots & M_1 & \vdots \\ \mathcal{O} & \cdots & \mathcal{O} & M_1 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} -\mathcal{I} & \mathcal{O} & \cdots & \mathcal{O} \\ \mathcal{O} & -\mathcal{I} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathcal{O} \\ \mathcal{O} & \cdots & \mathcal{O} & -\mathcal{I} \end{bmatrix},$$

$$\text{and } \rho_1 = \begin{bmatrix} \nu \\ \mu \\ \vdots \\ \mu \end{bmatrix}, \quad \rho_2 = \begin{bmatrix} s^* \\ u^* \\ \vdots \\ u^* \end{bmatrix}.$$

3 The feedback control approach

3.1 The optimal control framework

The general structure of the control problem can be represented as on Fig.2. Three requirements can be used to characterize the control problem :

1. stabilization in a region around the nominal conditions $x^* = 0$, corresponding to the nominal vector of stock levels s^*, and to the nominal vector of activity levels, u^*,

2. satisfaction of all the constraints along the system trajectory :

$$x_k \in R[Q,\rho], \ \forall k \in \mathcal{N}$$

3. Attenuation of the effect of the bounded persistent disturbance vector $w_k \in R[L,\lambda] \forall k \in \mathcal{N}$

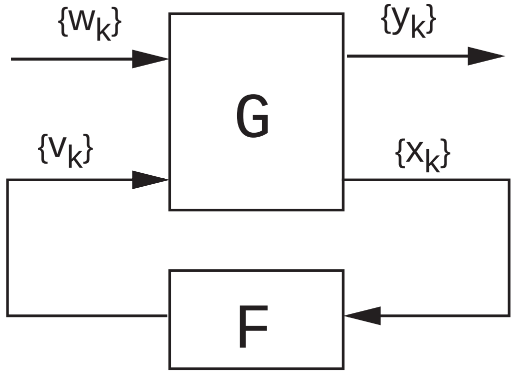

Figure 2: The control structure

Selecting the last requirement as the control problem objective, the problem can be interpreted as a constrained ℓ^∞ to ℓ^∞ (or ℓ^1) optimal control problem, and formulated as follows :

$$\begin{aligned} \text{Minimize }_{\{u_l\}} \|y\|_{\ell^\infty} = \sup_{k\in\mathcal{N}} \|y_k\|_\infty \\ \text{subject to } x_k \in R[Q,\rho],\ \forall k \in \mathcal{N} \\ x_{k+1} = \mathcal{A}x_k + \mathcal{B}_1 w_k + \mathcal{B}_2 v_k,\ \forall k \in \mathcal{N} \\ y_k = [I\ \mathcal{O}\ ...\ \mathcal{O}]x_{k+1},\ \forall k \in \mathcal{N} \\ w_k \in R[L,\lambda] \end{aligned}$$

In this version of the problem, it is supposed that the disturbance vector w_k belongs to a polyhedron $R[L,\lambda]$. It is a straightforward extension of the classical formulation of the ℓ^1 problem, in which the ℓ^∞ norm of the disturbance vector is supposed bounded. Several techniques have been proposed to solve standard versions of this problem (see e.g. [DP87], [BS95], [Sha96], [DH97] . As they are numerically involved, these techniques cannot be successfully applied to the high dimensional planning problems considered here. However, it is possible, even for large scale systems, to obtain feasible solutions providing reasonable upper bounds of the ℓ^1 criterion, using the positive invariance approach.

Firstly, to simplify the problem, only linear state feedback control laws will be investigated, in the form :

$$v_k = [F\ G_1\ \cdots\ G_\theta]x_k \tag{22}$$

The dynamic closed-loop state equation associated to a feedback (22) is :

$$x_{k+1} = \mathcal{A}x_k + \mathcal{D}w_k \tag{23}$$

with

$$\mathcal{A} = \begin{bmatrix} I+T_0F & T_1+T_0G_1 & \cdots & \cdots & T_\theta + T_0G_\theta \\ F & G_1 & \cdots & \cdots & G_\theta \\ \mathcal{O} & I & \mathcal{O} & \cdots & \mathcal{O} \\ \vdots & & & \ddots & \vdots \\ & \ddots & \ddots & & \\ \mathcal{O} & \cdots & \mathcal{O} & I & \mathcal{O} \end{bmatrix}, \text{ and } \mathcal{D} = \begin{bmatrix} I \\ \mathcal{O} \\ \vdots \\ \vdots \\ \\ \mathcal{O} \end{bmatrix}.$$

Moreover, if it is possible to construct in the state space a closed polytope Ω positively invariant with respect to system (23), then, application of the control sequence (22) to any trajectory starting in Ω guarantees the boundedness of all the components of x_k, $\forall k \in \mathcal{N}$ and provides an upper bound to $\sup_{k \in \mathcal{N}} \|y_k\|_\infty$.

3.2 The positive invariance approach

Positive invariance of a domain Ω with respect to system (23) means that any state trajectory starting in Ω remains in Ω, for any disturbance vector w_k in $R[L, \lambda]$.

The positive invariance approach is able to provide local solutions to constrained control problems by imposing the positive invariance of a domain Ω in the state space satisfying the following conditions [VHB88] :

1. The zero state lies in the interior of Ω,
2. Ω is positively invariant with respect to the controlled system,
3. $\Omega \subset R[Q, \rho]$, where $R[Q, \rho]]$ describes the polyhedron of constraints.

The polyhedron of constraints is defined by matrix Q and the positive vector ρ as in section 2.4.

3.3 The direct approach

Among the three conditions listed above, the first one and the third one are directly satisfied by the choice : $\Omega = R[Q, \rho]$. However, in general, it is not possible to satisfy the second condition for any polyhedron of constraints, specially when this polyhedron is compact, as it is the case here.

To test if a polyhedron $R[Q, \rho]$ can be made positively invariant with respect to system (23), it suffices to test the conditions of the following lemma (see e.g. [DH96]).

Lemma 1 *A necessary and sufficient condition for positive invariance of $R[Q, \rho]$ with respect to system (23) and for any disturbance vector w_k in $R[L, \lambda]$, is the existence of two nonnegative matrices H and J such that :*

$$HQ = Q\mathcal{A} \tag{24}$$
$$JL = Q\mathcal{D} \tag{25}$$
$$H\rho + J\lambda \leq \rho \tag{26}$$

This condition can be directly tested by Linear Programming. A possible objective function to be minimized is the contraction rate ϵ obtained by replacing condition (26) by :

$$H\rho + J\lambda \leq \epsilon\rho \tag{27}$$

Invariance conditions are satisfied if the optimal value satisfies

$$\epsilon^* \leq 1.$$

Then, the solution F, $G_1, ...,$, G_θ defines an feasible control law, through relation (22).

However, in general, the problem above may have a solution $\epsilon^* > 1$. In this case, the constrained problem may still be solved locally. Other possible choices for Ω can then be investigated, as shown in the following section.

3.4 The indirect approach

3.4.1 Sets of admissible initial conditions

The constrained control problem can be solved locally if it is possible to construct a polyhedron $\Omega = R[S, \sigma]$ such that :

1. $\sigma > 0$ (componentwise),
2. $R[S, \sigma]$ positively invariant,
3. $R[S, \sigma] \subset R[Q, \rho]$.

If the two first conditions are satisfied, the third condition can be replaced by the following condition (Extended Farkas' Lemma [Hen89]).

Lemma 2 *A necessary and sufficient condition to achieve $R[S, \sigma] \subset R[Q, \rho]$ is the existence of a nonnegative matrix P such that :*

$$PS = Q \tag{28}$$
$$P\sigma \leq \rho. \tag{29}$$

Clearly, if the conditions of Lemma 2 are satisfied, the domain $R[S, \sigma]$ defines a set of admissible initial conditions for the constrained control problem. However, limiting the initial conditions to any set $R[S, \sigma]$ satisfying the conditions of Lemma 2 would be unnecessarily restrictive. Rather, this set should be selected as "large" as possible. Several measures are available to compute the size of this domain, in particular its volume, if it is bounded. It is also possible to show the existence of a supremal positively invariant domain contained in $R[Q, \rho]$, But in general, this domain is not a polyhedron and is hard to compute (see [DH97] and references therein). A more practical way to tackle this problem is to solve the unconstrained control problem, and to construct the associated positively invariant domain contained in $R[Q, \rho]$ and which maximizes some particular size index. This approach will be developped in the next sections, using a particular control scheme which allows for an easy construction of polyhedral positively invariant domains.

3.4.2 Construction of a controlled invariant domain

The question of existence of positively invariant domains with respect to the considered system can be solved by the following theorem, proved in [HB98].

Theorem 1 *Positive invariance of certain compact polyhedral domains of the state space, $R[S, \sigma]$, can always be achieved with respect to the controlled system (23), under an appropriate choice of the gain matrices F, G_1, ..., G_θ of the control law (22).*

A candidate linear state feedback control law is defined by the following gain matrices :

$$\begin{aligned} F &= -(I - \Pi)^{-1}, \\ G_i &= -(I - \Pi)^{-1} E_i \text{ for } i = 1, ..., \theta. \end{aligned} \tag{30}$$

A class of associated positively invariant symmetrical domains is defined by :

$$S = \begin{bmatrix} D \\ -D \end{bmatrix} \text{ and } D = \begin{bmatrix} I & E_1 & \cdots & \cdots & E_\theta \\ \mathcal{O} & -(I-\Pi) & \mathcal{O} & \cdots & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & -(I-\Pi) & \ddots & \vdots \\ \vdots & & & \ddots & \mathcal{O} \\ \mathcal{O} & \cdots & & \mathcal{O} & -(I-\Pi) \end{bmatrix}$$

$$\sigma^T = [\delta^T \quad \delta^T] \text{ with } \delta^T = [\alpha^T \quad \beta^T \; ... \; \beta^T].$$

with α and β positive vectors in $\Re^N$ satisfying $\beta \le \alpha$.

By application of Lemma 1, positive invariance of $R[S,\sigma]$, for $\|\lambda\|$ sufficiently small, derives from the existence of matrix $\mathcal{H} = \begin{bmatrix} \mathcal{O} & \cdots & \cdots & \mathcal{O} \\ I & \mathcal{O} & \cdots & \vdots \\ \mathcal{O} & \ddots & \ddots & \\ \vdots & \ddots & & \\ \mathcal{O} & \cdots & I & \mathcal{O} \end{bmatrix}$ which satisfies

$$\mathcal{H}D = D\mathcal{A}$$
$$\mathcal{H}\delta \le \delta$$

and from the choice of positive vectors α and $\beta \le \alpha$ in $\Re^N$, under the condition :

$$\alpha_i \ge \max(\overline{\omega}_i, \underline{\omega_i}) \; \forall i = 1, ..., N. \tag{31}$$

Conditions (1) and (2) of section 3.4.1 are satisfied by the control law defined by (30). Condition (3) is also satisfied if the conditions of Lemma 2 apply. These conditions impose constraints on the choice of vector δ such that $\sigma^T = [\delta^T \quad \delta^T]$. As matrix P and vector δ are both unknown, inequality (29) is nonlinear, and the best possible choice of vector δ depends upon the selected criterion. A practical technique for satisfying the conditions of Lemma 2 is to start from an arbitrary choice of vector δ_0, then to find a matrix P satisfying (28) and (29), and to optimize the choice of vector δ for this particular matrix P. If the optimized vector δ satisfies conditions (31), the domain $R[S,\sigma]$ is admissible. If not, the problem can be reversed by computing, from the same conditions (31), the maximal levels of disturbances $(\overline{\omega}_i, \underline{\omega_i})$ which can be tolerated for each product i.

Finally, if the conditions of Lemma 2 are satisfied by the domain $R[S,\sigma]$, this domain defines a set of admissible initial conditions from which a feasible production trajectory is generated for the closed-loop planning problem, under the closed-form control law defined by the gain matrices in (30). The disturbance attenuation properties of the selected control law can be evaluated by noting that due to the symmetry property of $R[S,\sigma]$, the inclusion $R[S,\sigma] \subset R[Q,\rho]$ implies

$$|y_k| \le s^* \quad \forall k \in \mathcal{N}, \forall \{\omega_l\} | \omega_l \in R[L,\lambda], \forall l \in \mathcal{N}.$$

4 Conclusion

Integration of production lead-times in the input-output model of a multistage production system has led to a discrete-time dynamic model with activity levels as control variables, stock levels as output variables and additive disturbances representing demand and production fluctuations. A state representation of the system has been constructed and the multistage planning problem has been formulated as a state feedback control problem. As the system is also subject to many linear constraints on its control and output variables, an efficient approach to solve the control problem is based on positive invariance of polyhedral sets. Linear matrix conditions have been established to characterize candidate linear control laws and their associated positively invariant domains. An explicit state feedback control law has been proposed and associated polyhedral positively invariant domains have been constructed. The conditions of inclusion of such a polyhedron in the domain of constraints then guarantee the feasibility of the plan.

References

[Bak93] K.R. Baker. Requirements planning. In *Handbooks in Operations Research and Management Science,* **vol. 4**, *S.C. Graves, A.H.G. Rinnooy Kan, P.H. Zipkin Eds.*, pages 571–628. North-Holland, 1993.

[BP79] A. Berman and R.J. Plemmons. *Non Negative Matrices in the Mathematical Sciences.* Academic Press, 1979.

[BS95] F. Blanchini and M. Sznaier. Persistent disturbance rejection via static-state feedback. *IEEE Trans. Automat. Contr.*, 40(6):1127–1131, 1995.

[Bur89] J.L. Burbidge. *Production flow analysis for planning group technology.* Oxford Science Publications, 1989.

[DA87] R. David and H. Alla. Continuous petri nets. In *Proceedings 8th European Conf. on Application and Theory of Petri nets, Saragoza, Spain*, pages 275–294, 1987.

[DH96] C. E. T. Dórea and J.-C. Hennet. On (a,b)-invariance of polyhedral domains for discrete-time systems. In *Proceedings of the 35th IEEE Conference on Decision and Control*, pages 4319–4324, Kobe, Japan, 1996.

[DH97] C. E. T. Dórea and J.-C. Hennet. A geometric approach to the l^1 linear control problem. In *36th IEEE Conference on Decision and Control*, volume 2, pages 1552–1557, San Diego, California, USA, 1997.

[DP87] M. A. Dahleh and J. B. Pearson. l^1-optimal feedback controllers for mimo discrete-time systems. *IEEE Trans. Automat. Contr.*, 32(4):314–322, 1987.

[FH83] D.W. Fogarty and T.R. Hoffmann. *Production and inventory management.* South-Western Publishing Company, 1983.

[GM94] R.W. Grubbström and A. Molinder. Material requirement planning employing input-output analysis and laplace transforms. *Intl. J. Production Economics,* **vol. 35**, *pp.299-311*, 1994.

[GO92] R.W. Grubbström and P. Ovrin. Intertemporal generalization of the relationship between material requirement planning and input-output analysis. *Intl. J. Production Economics,* **vol. 26**, *pp.311-318*, 1992.

[HB98] J.C. Hennet and I. Barthès. Closed-loop planning of multi-level production under resource constraints. In *Proceedings of the IFAC Symposium INCOM'98, Nancy, (France)*, 1998.

[Hen89] J-C. Hennet. Une extension du lemme de farkas et son application au problème de régulation linéaire sous contraintes. *C.R. Acad. Sciences,* **t.308***,* **Série I***, pp.415-419*, 1989.

[HV87] J.C. Hennet and M. Vassilaki. Modélisation et gestion de systèmes de production avec stockages. *RAIRO-APII,* **Vol. 21***, pp.3-16*, 1987.

[Kar93] U.S. Karmarkar. Manufacturing lead-times, order release and capacity loading. In *Handbooks in Operations Research and Management Science,* **vol. 4***, S.C. Graves, A.H.G. Rinnooy Kan, P.H. Zipkin Eds.*, pages 287–329. North-Holland, 1993.

[KWW75] G.J. Kohler, A.B. Whinston, and G.P. Wright. *Optimization over Leontief substitution systems.* North Holland / American Elsevier, 1975.

[Mer76] P. Merlin. Methodology for the design and implementation of communication protocols. *IEEE Trans. Communication,* **vol. 24***,* **no. 6**, 1976.

[Mol95] A. Molinder. *Further theoretical considerations on the relationship between MRP, Input-Output analysis and multi-echelon inventory systems.* PROFIL (Production-Economic Research in Linköping), 1995.

[Sha96] J. S. Shamma. Optimization of the l^{∞}-induced norm under full state feedback. *IEEE Trans. Automat. Contr.*, 41(4):533–544, 1996.

[Vaz55] A. Vazsonyi. The use of mathematics in production and inventory control. *Management Science,* **Vol. 1***,* **No. 1** *pp.70-85*, 1955.

[Vei69] A.F.Jr Veinott. Minimum concave-cost solution of leontief substitution models of multi-facility inventory systems. *Operations Research,* **Vol.17, No 2***, pp.262-291*, 1969.

[VH86] M. Vassilaki and J.C. Hennet. Modelling and safety stock analysis in a multi-product industrial process. In *Proceedings of the IMACS-IFAC Symposium, Lille, France, pp.439-442*, 1986.

[VHB88] M. Vassilaki, J.C. Hennet, and G. Bitsoris. Feedback control of linear discrete-time systems under state and control constraints. *Int. Journal of Control,* **vol 47** *, pp.1727-35*, 1988.