

SRAFFA AND LEONTIEF ON JOINT PRODUCTION

by

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Abstract

We show that results obtained in the Sraffian literature on joint production are relevant for input-output economists working in the tradition of Leontief. We concentrate upon the ‘adjustment’ property, i.e. the ability of a system to cope with any variation in final demand. We identify a number of systems which possess the adjustment property, with special attention for economies using fixed capital goods.

1. SRAFFA AND LEONTIEF: DIFFERENCES IN AIMS AND FORMALIZATIONS

For both Sraffa and Leontief, production constitutes the heart of the economic system. They both conceive it as a circular process, in which commodities are reproduced by means of commodities and labour. Yet in spite of this common basic framework, there has been little, if any, contact between the Sraffa-inspired and the Leontief-inspired approaches. (One notable exception is Pasinetti, 1977.) The gap between the two constructions reflects a difference in objectives: roughly speaking, Sraffa's aim is mainly theoretical and oriented towards the study of prices and distribution, while Leontief's description of the structure of an economy is mainly empirical and oriented towards economic policy.

Whatever the ultimate objective, it is natural to begin the analysis of a linear model of production with the case of single-product systems. The core of any linear model of production, be it one of the Leontief (1951), von Neumann (1945-46), or Sraffa (1960) variety, consists of a pair of matrices (\mathbf{A} , \mathbf{B}), where \mathbf{A} represents inputs and \mathbf{B} outputs.¹ When every process produces exactly one commodity, the output matrix can be reduced to an identity matrix after a convenient choice of physical units. In that case all the relevant information is contained in the so-called 'input-output' matrix \mathbf{A} , which is in fact an input matrix only.

A crucial property of single-product systems is that they can cope with any variation in final demand, provided that the activity levels of the processes already in operation are appropriately chosen. We will say that a system with this characteristic has the adjustment property: as soon as it is able to produce *some* positive net product, it can produce *any* semipositive net product. That property is intimately connected with the positivity of the Leontief-inverse matrix $(\mathbf{I} - \mathbf{A})^{-1}$.

The divergence between Sraffa's and Leontief's formalizations comes to the surface as soon as one considers multiple-product systems, *i.e.* systems in which at least one process produces several goods simultaneously. From a theoretical point of view, the only adequate formalization consists in using an output matrix \mathbf{B} which admits several positive elements in at least one of its columns. In the literature based upon von Neumann's or Sraffa's work, this is the standard practice. The input-output (I-O) specialists working in the tradition of Leontief, however, stick to a representation in terms of a modified input-output matrix $\bar{\mathbf{A}}$ (*cf.* Miller & Blair, 1985, for a survey of different possibilities, and Kop Jansen & Ten Raa, 1990, and Konijn & Steenge, 1995, for recent contributions in this field).

The central question addressed in this paper is: when is it possible to replace the 'extensive form' (\mathbf{A}, \mathbf{B}) by a 'reduced form' $(\bar{\mathbf{A}}, \mathbf{I})$? From a theoretical point of view, the answer is: never! A single-product system cannot mimic the behaviour of a multiple-product system in all respects, however small the divergence from single production is. We do understand, however, the great advantage of simplicity introduced by approximating the true data (\mathbf{A}, \mathbf{B}) by a proxy $(\bar{\mathbf{A}}, \mathbf{I})$. In order to justify that current practice, one might for instance try to estimate the distortions it introduces.

The path followed in this work is slightly different. We start from Sraffa's theoretical approach, and show that Sraffian analysis sheds new light on the problem. We analyze a multiple-product system in its extensive form and wonder if it behaves like a single-product system in some respects. The property we stress is the flexibility of the economy with respect to changes in quantities. In other words, the question is to identify some specific types of multiple-product systems which still have the adjustment property mentioned above. They are characterized by the fact that the inverse matrix $(\mathbf{B} - \mathbf{A})^{-1}$ is positive (or semipositive, in case of

decomposability) and are called ‘all-engaging’ (respectively, ‘all-productive’) systems. They will be studied in Section 3.

The simplest case in which the adjustment property is met is when every process admits a main product. Although the output matrix is then not diagonal, it is nevertheless a matrix of the dominant diagonal type, and the nature of each process is directly determined by its main product. The matter becomes more complex when some processes do not have a main product, especially if we introduce the possibility of choice among processes. We will say that a system possesses the super-adjustment property if every viable selection of processes has the adjustment property. Interestingly, such a system happens to be a non-substitution economy. One way of identifying systems having the super-adjustment property consists in classifying processes which compete with one another (*e.g.* because they produce the same dominant product) in the same industry, more precisely in the same sector. Section 4 defines the general concept of sector without referring to dominant products.

In the presence of pure capital goods, *i.e.* goods which are demanded only because they are indispensable for the production of other commodities, the adjustment property must be modified: it means that the economic system can adapt itself to any semipositive final demand vector, knowing that the components of that vector corresponding to pure capital goods are identically zero (Section 5).

An alternative way to treat the same problem consists in eliminating the pure capital goods from the economy by ‘integrating’ the processes in such a way that the integrated processes have automatically a zero net product of pure capital goods. Intuitively speaking, the adjustment property is expected to hold for the economy under two conditions: first, the integration, *i.e.* the procedure which eliminates the purely intermediate goods, is smooth

enough; and second, the integrated economy, which is obtained after elimination of the capital goods, has the adjustment property. Section 6 examines the question in more details.

The identification of the different types of multiple-product systems which, like single-product systems, possess the adjustment property, is the result of the endeavours of a number of Sraffian economists. Their aim is to explore the path indicated by Sraffa. They do not hesitate to criticize the treatment of joint production in I-O analysis, especially that of fixed capital. As we show in this paper, however, their critique also has a constructive aspect: the results upon which we report here are useful for economists interested in the Sraffa-approach as well as for those specialized in I-O analysis. As a by-product of their activity, so to speak, Sraffians make an important ‘outside’ contribution to the foundations of I-O analysis. In our view this exemplifies the vitality of the Sraffian approach. Our paper should be seen as an attempt to prompt a dialogue between economists of both ‘camps’ based on their common conception of production.

2. THEORETICAL BACKGROUND AND NOTATION

We consider an economy in which m different production processes are available to produce n commodities. All processes operate under constant returns to scale. Production process i ($i = 1, \dots, m$) is described by a nonnegative $n \times 1$ vector of commodity inputs \mathbf{a}_i , a semipositive $n \times 1$ vector of commodity outputs \mathbf{b}_i , and a labour input. In the present study the labour inputs are ignored. The activity level of process i is denoted as y_i , a nonnegative scalar. For the economy as a whole, the $n \times m$ matrix \mathbf{A} represents the m input vectors \mathbf{a}_i , the $n \times m$ matrix \mathbf{B} the m output vectors \mathbf{b}_i , and the $m \times 1$ vector \mathbf{y} the m activity levels y_i . The couplet (\mathbf{A}, \mathbf{B}) is called a ‘system of production’ or ‘system’.

If $m = n$ (the numbers of processes and commodities are equal) the system is square. The matrices \mathbf{A} and \mathbf{B} are semipositive, and matrix \mathbf{B} must have at least one positive element in each row if all commodities are to be reproduced. A single-product process has all its output coefficients b_{ij} equal to zero, but one. By the constant returns assumption, the non-zero coefficient can be set equal to one. Hence a system which can reproduce itself, consists of single-product processes only, and is square, is represented by the pair (\mathbf{A}, \mathbf{I}) or, more briefly, by the input matrix of technical coefficients \mathbf{A} .

A square system is called non basic (or decomposable, or reducible) when matrices \mathbf{A} and \mathbf{B} can be written as:

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{IJ} & \mathbf{0} \\ \mathbf{A}_{\bar{I}J} & \mathbf{A}_{\bar{I}\bar{J}} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{B}_{IJ} & \mathbf{0} \\ \mathbf{B}_{\bar{I}J} & \mathbf{B}_{\bar{I}\bar{J}} \end{bmatrix} \quad \text{with } \text{card } \bar{J} = \text{card } \bar{I}.$$

In this case the commodities belonging to \bar{I} reproduce themselves by means of the processes in \bar{J} . In other words, there exists a sub-economy which works autonomously. For the sake of simplicity, we exclude this case most of the time, *i.e.* the economy is assumed to be *basic* (Sraffa's terminology) or indecomposable, or irreducible. The question is discussed in Section 3.

We build upon results which belong to both the Leontief-oriented and the Sraffa-oriented literature. The concept of all-engaging system comes from the Sraffian tradition: it was introduced by Schefold (1971), in a Ph.D. dissertation entitled *Mr. Sraffa on Joint Production*, and refined in Schefold (1978). The notion of a non-substitution economy, on the other hand, was developed in papers by Arrow (1951), Georgescu-Roegen (1951) and Samuelson (1951) dealing with Leontief's input-output model. With regard to non-substitution in the presence of fixed capital our results should be situated in a line of contributions starting with Samuelson (1961), Mirrlees (1969), Stiglitz (1970), Salvadori (1988) and Bidard (1996b). We firmly

believe, however, that our results are of interest for I-O economists, too. Directly or indirectly the question we address here concerns the generalization of the Leontief-inverse $(I - A)^{-1}$.

Sraffa and the Sraffians are mainly concerned with the properties of the price vector which ensures a uniform profit rate among all industries. From a formal point of view, any price equation can be associated with a ‘dual’ quantity equation obtained by transposition. In this operation the price vector becomes a vector of activity levels, the labour vector is transformed into a final demand vector, and the profit rate is read as a growth rate. For instance the Ricardian trade-off property, which holds true in single production, states that the real wage and the profit rate vary in opposite directions. Once reinterpreted by duality, the property is read: for a given final demand, all activity levels increase if the rate of accumulation increases. Such reinterpretations facilitate the dialogue with economists specialized in I-O analysis. Sections 3 to 7 state a certain number of useful results for economists working in one or the other field. No proofs are given, but the interested reader will find the references to the original papers.

3. ALL-ENGAGING SYSTEMS

3.1. The adjustment property

In this and the next Section we assume that all commodities may be consumed (the case in which there is no final demand for some goods will be examined in Section 5). The notion of a *productive* (or *viable*) system is the same for single- or multiple-product systems. Let us give it in the general case:

Definition 1. A technique (\mathbf{A}, \mathbf{B}) is said to be (strictly) *viable* if it can produce a physical net surplus of any commodity. Formally, there is an activity vector \mathbf{y} such that:²

$$\exists \mathbf{y} \geq \mathbf{0}, \quad \mathbf{B}\mathbf{y} - \mathbf{A}\mathbf{y} > \mathbf{0} \quad (1)$$

A system is productive when the coefficients of the input matrix \mathbf{A} are small enough. The productivity hypothesis is a weak condition: nonproductive economics could not reproduce themselves and are not observable. The property we are mainly interested in is:

Definition 2. A technique has the *adjustment property* (or is *adjustable*) if it can produce any semipositive net final demand vector. Formally:

$$\forall \mathbf{d} \geq \mathbf{0}, \quad \exists \mathbf{y} \geq \mathbf{0}, \quad \mathbf{B}\mathbf{y} - \mathbf{A}\mathbf{y} = \mathbf{d} \quad (2)$$

We begin by assuming that the system is square, *i.e.* it contains as many processes as commodities; in Section 4 we abandon this restrictive assumption. In joint production, the activity levels \mathbf{y} solution to (2) are $\mathbf{y} = (\mathbf{B} - \mathbf{A})^{-1}\mathbf{d}$. This formula is economically acceptable if the activity levels are semipositive, and the way to ensure this is the semipositivity of the inverse matrix $(\mathbf{B} - \mathbf{A})^{-1}$. The difference between single and joint production is that, in the first case, the inverse matrix $(\mathbf{I} - \mathbf{A})^{-1}$ is automatically semipositive as soon as the input-output matrix \mathbf{A} is productive. No similar property holds in general joint production.

Theorem 1. The square system (\mathbf{A}, \mathbf{B}) has the adjustment property if and only if matrix $(\mathbf{B} - \mathbf{A})$ is regular and admits a semipositive inverse.

For the sake of simplicity, we first consider the slightly more restrictive condition that the inverse matrix is in fact positive.

3.2. Characterizations of all-engaging systems

Definition 3. The square system (A, B) is *all-engaging* if matrix $(B - A)$ is regular and admits a positive inverse.

An equivalent formulation of all-engagingness which is very convenient for economic analysis is:

Theorem 2. The square system (A, B) is all-engaging if and only if properties (3) and (4) hold:

$$\exists y_0 \geq 0, \quad (B - A)y_0 \geq 0 \quad (3)$$

$$\{y \geq 0, (B - A)y \geq 0\} \Rightarrow y > 0 \quad (4)$$

Relation (3) is the ‘weak’ productivity (or viability) hypothesis. Relationship (4) is read: in order to obtain *some* net product all processes *must* be operated. It is not obvious that Definition 3 and Theorem 2 are indeed equivalent.

An all-engaging system retains many economic properties of single-product economies. For instance, the adjustment property to final demand still holds if the rate of accumulation g is positive and not too high. The notion of all-engagingness is easily generalised to g -all-engagingness.

Definition 4. Let g ($g > -1$) represent a rate of accumulation. The square system (A, B) has the *g-adjustment property* if it can produce any final demand vector d after accumulation at rate g .

An all-engaging system is g -all-engaging at $g = 0$. Conversely, let us replace matrix A by $(1+g)A$ in relations (3) and (4). Theorem 2 then gives a characterization of g -all-engaging systems. The specific interest in the generalization of the notion of all-engagingness to that of g -all-engagingness is shown by the following result:

Theorem 3. If (A, B) is g -all-engaging for some value g , the set S' on which it remains g -all-engaging is an interval $S' =]g_{\min} = g_0, g_{\max} = G[$. The upper bound of S' is characterized by the properties:

$$\exists! \mathbf{q} > \mathbf{0}, \quad [\mathbf{B} - (1+G)\mathbf{A}]\mathbf{q} = \mathbf{0} \quad (5)$$

$$\nexists \mathbf{y} \geq \mathbf{0}, \quad [\mathbf{B} - (1+G)\mathbf{A}]\mathbf{y} \geq \mathbf{0} \quad (6)$$

(In relation (5) symbol $\exists! \mathbf{q}$ indicates the existence and uniqueness of activity levels \mathbf{q} , up to a positive scalar.)

Theorem 3 shows that an all-engaging system remains g -all-engaging for $g > 0$, provided that g is smaller than the upper bound G . Value G is the maximal accumulation rate, as defined by von Neumann (1945-46). Relationship (5) means that there exist positive activity levels such that the economy moves on a regular growth path at the maximal growth rate G , and these activity levels are unique (up to a positive factor). Relation (6) means that no surplus can be obtained at rate G , whatever the activity levels. Thus scalar G appears as the maximal feasible rate, and \mathbf{q} as the von Neumann activity levels for system (A, B) .³

In basic single production, when \mathbf{A} is a semipositive indecomposable input matrix, matrix $(\mathbf{I} - (1+g)\mathbf{A})^{-1}$ is positive for any g smaller than G , where $\Lambda = (1+G)^{-1}$ is the Perron-Frobenius eigenvalue of \mathbf{A} . The Perron-Frobenius eigenvectors are positive and unique up to a factor of proportion. Theorem 3 characterises g -all-engaging systems with similar properties. It must however be noted that in joint production the value G is a root of $\det(\mathbf{B} - (1+g)\mathbf{A}) = 0$, but not necessarily the first positive root (which is the case in single production, as a consequence of the fact that the Perron-Frobenius root of a matrix is the maximal one; *cf.* Sraffa, 1960, §42).

Up to now we have considered an all-engaging system and, by increasing g , we have obtained g -all-engaging systems until g reaches a maximal value G which has specific properties. Let us now look at the same problem the other way around, *i.e.* we start from the upper bound G and try to reach the value $g = 0$ by decreasing g continuously. If the g -all-engagingness property is preserved during this operation, then system (\mathbf{A}, \mathbf{B}) is all-engaging.

The operation can indeed be performed for viable basic single-production system. In a left neighbourhood of the maximal growth rate G , defined by $1+G = 1/\Lambda$ as indicated above, and for which conditions (5) and (6) hold, the system is indeed g -all-engaging, and this neighbourhood contains the value $g = 0$.

The first difference due to joint production is that the existence of a value G such that conditions (5) and (6) are met is not guaranteed. In that case, the system cannot be all-engaging. Assume the existence of such a value. Then it can be shown that (\mathbf{A}, \mathbf{B}) is indeed g -all-engaging on some left neighbourhood of G . The second difference is that this neighbourhood does not necessarily contain value $g = 0$. The system is all-engaging when, in terms of the set S' described in Theorem 3, we have $g_0 < 0$. To sum up, all-engagingness depends on the existence of a value G satisfying (5) and (6), and the fact that g_0 is low enough. While the first necessary condition can easily be checked by studying the spectral properties of

(\mathbf{A}, \mathbf{B}) , there does not exist a simple rule to check that g_0 is indeed negative (a simple formula for a lower bound for g_0 can be found, but what is expected in the present case is an upper bound).

3.3. All-productive systems

We now turn to a slightly more general notion than all-engagingness:

Definition 5. The square system (\mathbf{A}, \mathbf{B}) is *all-productive* if matrix $(\mathbf{B} - \mathbf{A})$ is regular and admits a semipositive inverse.

According to Theorem 1, the adjustment property holds true for square systems if and only if the system is all-productive. The modified version of Theorem 2 now reads:

Theorem 4. The square system (\mathbf{A}, \mathbf{B}) is all-productive if and only if properties (7) and (8) hold:

$$\exists \mathbf{y}_0 \geq \mathbf{0}, \quad (\mathbf{B} - \mathbf{A})\mathbf{y}_0 > \mathbf{0} \tag{7}$$

$$\{\mathbf{y} \geq \mathbf{0}, (\mathbf{B} - \mathbf{A})\mathbf{y} > \mathbf{0}\} \Rightarrow \mathbf{y} > \mathbf{0} \tag{8}$$

In single production a viable square system (\mathbf{A}, \mathbf{I}) is automatically all-productive, and this is why the adjustment property then holds without restriction. It is moreover all-engaging if and only if it is indecomposable, *i.e.* if there does not exist a proper subset of commodities and processes which work autonomously.

In joint production a viable square system (\mathbf{A}, \mathbf{B}) is not always all-productive. But, if it is, the distinction between all-productive systems and all-engaging systems depends similarly on the fact that (\mathbf{A}, \mathbf{B}) is decomposable or not, with one qualification. A numerical example with two commodities and two processes illustrates the exception.

Example 1. Consider the system:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}.$$

(\mathbf{A}, \mathbf{B}) is g -all-engaging for g in $S' =]0, G = (5 - \sqrt{21})/2 [$. For $g = 0$, we have:

$$(\mathbf{B} - \mathbf{A})^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

This example illustrates the only case where an irreducible system (\mathbf{A}, \mathbf{B}) is all-productive but not all-engaging:

Theorem 5. If (\mathbf{A}, \mathbf{B}) is all-productive but not all-engaging:

- either (\mathbf{A}, \mathbf{B}) is irreducible and 0 is the lower bound g_0 of the interval S' defined in Theorem 3;
- or (\mathbf{A}, \mathbf{B}) is reducible, and the set on which (\mathbf{A}, \mathbf{B}) is g -all-productive is an interval $S = [g_0, G[$ which contains value $g = 0$. The upper bound G of S has the two properties:

$$\exists \mathbf{q} \geq \mathbf{0}, \quad [\mathbf{B} - (1 + G)\mathbf{A}]\mathbf{q} = \mathbf{0} \tag{9}$$

$$\exists \mathbf{y} \geq \mathbf{0}, \quad [\mathbf{B} - (1 + G)\mathbf{A}]\mathbf{y} > \mathbf{0} \tag{10}$$

If relationships (5) and (6) are satisfied, then (A, B) is g -all-engaging on some left neighbourhood of G . On the contrary, the existence of G satisfying (9) and (10) does not guarantee that S is nonempty. This difference is a reason to give some privilege to the notion of all-engaging system.

Sraffa's readers may have been surprised by the notion of decomposability we consider. Though it is an immediate extension of the notion used for single-product systems, it is not the one used by Sraffa himself. What happens is that, in joint production, there does not exist an all-encompassing notion of decomposability: Sraffa's concept of a non-basic system is indeed relevant for the type of economic effect he contemplates (the incidence of taxation on prices; *cf.* Sraffa, 1960, § 65), but less so for the study of the adjustment property.

3.4. Comparison with the I-O approach

In both the Sraffian and the I-O literature a basic idea for the analysis of multiple-product systems is to identify cases in which some properties of single-product systems still hold. Perhaps the most important property in this regard is the capability of the economic system to adapt itself to a given final demand vector. Let us briefly compare the Sraffian and the I-O approaches to the problem, assuming the system is square.

As explained above, Sraffians start from the equation:

$$By = Ay + d \quad (11)$$

Observing that the solution of this equation is:

$$y = (B - A)^{-1}d \quad (12)$$

they are especially interested in finding conditions which ensure the semipositivity of the inverse matrix $(B - A)^{-1}$.

I-O economists, on the other hand, start from the following relationship (*cf.* Kop Jansen & Ten Raa, 1990; Steenge, 1990):

$$\mathbf{V}^T \mathbf{e} = \mathbf{U} \mathbf{e} + \mathbf{f} \quad (13)$$

which is closely related to equation (11). Equation (13) is of course expressed in money terms, \mathbf{V}^T being the make-matrix (counterpart of matrix \mathbf{B}), \mathbf{U} the use-matrix (counterpart of matrix \mathbf{A}), and \mathbf{f} the vector of final demand (counterpart of vector \mathbf{d}). The equation is meant as a description of a given economic system, in which case the vector \mathbf{e} represents the effective activity levels of the different industries (all equal to one by convention). I-O economists have tried to transform expression (13) into the following pseudo-single-product system equation:

$$\mathbf{x} = \bar{\mathbf{A}} \mathbf{x} + \mathbf{f} \quad (14)$$

with $\mathbf{x} = \mathbf{V}^T \mathbf{e}$ and $\bar{\mathbf{A}}$ representing a ‘surrogate’ input-output matrix. This leads to material balance equation:

$$\bar{\mathbf{A}} \mathbf{V}^T \mathbf{e} = \mathbf{U} \mathbf{e} \quad (15)$$

expressing that the inputs required to produce to the total output $\mathbf{V}^T \mathbf{e}$ are equal to the available inputs. In the I-O literature many ways have been suggested to define the input-output matrix $\bar{\mathbf{A}}$. A consensus seems to have been reached around the solution based upon the so-called ‘commodity-technology’ assumption, which leads to the following definition:

$$\bar{\mathbf{A}} = \mathbf{U}(\mathbf{V}^T)^{-1} \quad (16)$$

The total output vector \mathbf{x} corresponding to the final demand vector \mathbf{f} is therefore given by:

$$\mathbf{x} = (\mathbf{I} - \bar{\mathbf{A}})^{-1} \mathbf{f} = \left[\mathbf{I} - \mathbf{U}(\mathbf{V}^T)^{-1} \right]^{-1} \mathbf{f} \quad (17)$$

which explains why many I-O economists are particularly interested in determining whether matrix $\mathbf{U}(\mathbf{V}^T)^{-1}$ is semipositive or not.

Let us now compare the properties of both the Sraffian and the the I-O solutions. To simplify things, we assume that the physical units in which \mathbf{A} , \mathbf{B} and \mathbf{d} are expressed are chosen in such a way that the prices of all goods are equal to one.⁴ Under these circumstances, it is easy to see that we have:

$$\mathbf{V}^T = \mathbf{B}\hat{\mathbf{Y}}, \quad \mathbf{U} = \mathbf{A}\hat{\mathbf{Y}}, \quad \mathbf{f} = \mathbf{d}, \quad \mathbf{y} = \hat{\mathbf{Y}}\mathbf{e} \quad (18)$$

where $\hat{\mathbf{Y}}$ represents a diagonal matrix with the elements of vector \mathbf{y} on the main diagonal. Relations (18) allows us to establish a link between systems (11) and (13).

In the Sraffian framework, the adjustment property holds if and only if $(\mathbf{B} - \mathbf{A})^{-1} \geq \mathbf{0}$. Using expressions (18), it can be shown that the equivalent condition in I-O notation is $(\mathbf{V}^T - \mathbf{U})^{-1} \geq \mathbf{0}$. One may wonder whether this condition translates into a similar property of the generalized Leontief-inverse matrix $[\mathbf{I} - \mathbf{U}(\mathbf{V}^T)^{-1}]^{-1}$. The following identities are easily established:

$$(\mathbf{V}^T - \mathbf{U})^{-1} = (\mathbf{V}^T)^{-1}[\mathbf{I} - \mathbf{U}(\mathbf{V}^T)^{-1}]^{-1} \quad (19)$$

$$[\mathbf{I} - \mathbf{U}(\mathbf{V}^T)^{-1}]^{-1} = \mathbf{V}^T(\mathbf{V}^T - \mathbf{U})^{-1} \quad (20)$$

Therefore:

1. If the adjustment property holds, *i.e.* if $(\mathbf{V}^T - \mathbf{U})^{-1} \geq \mathbf{0}$, the generalized Leontief-inverse matrix is also semipositive (this can be inferred from (20), given that $\mathbf{V}^T \geq \mathbf{0}$). Note, however, that the constructed input-output matrix $\mathbf{U}(\mathbf{V}^T)^{-1}$ need not be semipositive.
2. If the generalized Leontief-inverse matrix is semipositive, *i.e.* if $[\mathbf{I} - \mathbf{U}(\mathbf{V}^T)^{-1}]^{-1} \geq \mathbf{0}$, the matrix $(\mathbf{V}^T - \mathbf{U})^{-1}$ need not be semipositive, as can be deduced from (19). Hence it may be that the system is not capable to adapt itself to an arbitrary final demand vector.
3. If the constructed input-output matrix is semipositive, *i.e.* if $\mathbf{U}(\mathbf{V}^T)^{-1} \geq \mathbf{0}$, the matrix $(\mathbf{V}^T - \mathbf{U})^{-1}$ again need not be semipositive; hence the adjustment may not be satisfied.

In conclusion, it is our opinion that the importance attributed by I-O specialists to the semipositivity of the constructed input-output matrix $U(\mathbf{V}^T)^{-1}$ or of the generalized Leontief-inverse matrix $[\mathbf{I} - U(\mathbf{V}^T)^{-1}]^{-1}$, is somewhat misplaced. It appears that more attention should be given to the semipositivity of matrix $(\mathbf{V}^T - \mathbf{U})^{-1}$.

4. SUPER-ADJUSTMENT AND SECTORS

4.1. Choice of techniques and super-adjustment

Typical for both all-engaging and all-productive systems is that all available processes will be activated as soon as a strictly positive net output vector must be produced. This is clearly a limiting case: in more realistic settings there will be much more processes than will be used effectively. Usually some kind of rule is specified according to which a choice will be made among the existing processes (*e.g.* cost minimization at a given rate of profits). This is the domain of the theory of ‘choice of techniques’; we deal with one aspect of it here.

To distinguish the cases in which a choice has to be made from those in which no choice can be made, we first introduce a convenient notation. Let $M = \{1, 2, \dots, m\}$ be the set of available processes in system (\mathbf{A}, \mathbf{B}) . Given a subset M_α of M , the technique denoted $(\mathbf{A}_\alpha, \mathbf{B}_\alpha)$ is the system in which the set of available processes is restricted to M_α .

We start from the notion of minimality:

Definition 6. The system (\mathbf{A}, \mathbf{B}) is *minimal* if it is viable and none of its techniques $(\mathbf{A}_\alpha, \mathbf{B}_\alpha)$ is viable.

All-engaging and all-productive systems are minimal. This can be deduced from the following fundamental result:

Theorem 6. If a multiple-product system has any two of the following three properties:

- (i) it is minimal;
- (ii) it is square;
- (iii) it is adjustable;

it has the third one. It is then *all-productive*.

Things are more complicated for non-minimal systems. The issue at stake is not so much whether the system is adjustable, but whether the technique (or techniques) which will be chosen is adjustable. Here we do not specify the rule which governs the choice of techniques, but simply assume that whatever it is, it must lead to the selection of one or more *viable* techniques. Under that assumption, it is of interest to know whether or not all viable techniques are adjustable. This is what we mean by super-adjustment:

Definition 7. The system (\mathbf{A}, \mathbf{B}) has the *super-adjustment property* if every technique $(\mathbf{A}_\alpha, \mathbf{B}_\alpha)$ which is viable is also adjustable.

4.2. Sectors

Let us reconsider the case of all-productive systems. A simple case of an all-productive square system (\mathbf{A}, \mathbf{B}) occurs when matrix $(\mathbf{B} - \mathbf{A})$ has non-positive off-diagonal coefficients and a

positive diagonal with great enough coefficients. In economic terms, the i -th process has a main product i and eventually a series of by-products, the production of which does not exceed the quantity required as input. In single production (no by-products), it is said that the process belongs to industry i ; in joint production, it belongs to sector i .

The existence of a main product in the sense defined above is a very stringent hypothesis, so restrictive that it is almost useless. When dealing with the question of choice of techniques, economists, and specialists of input-output analysis in particular, have used the notion of sector in a broader sense, which is however not well-specified. Here we lay the conceptual bases for a theoretically relevant definition of the concept of sector. It is based on two ideas:

- (i) What is important is not the distribution of signs in matrix $(\mathbf{B} - \mathbf{A})$, but the fact that $(\mathbf{B} - \mathbf{A})$ admits a semipositive inverse.
- (ii) In single production, the division into industries is based on a natural partition of the processes according to the commodity produced. The selection of operated methods leads most probably to the choice of one process in every industry. In other words, all processes within industry i compete with each other, and the final technique is obtained by picking exactly one method from each industry.

A similar structure is described in Definition 8, without the dominant product hypothesis.

Definition 8. Let there be n commodities and m processes, $m \geq n$. The system (\mathbf{A}, \mathbf{B}) is a *sectoral economy* if it is possible to partition the set of processes M into n sectors I_1, I_2, \dots, I_n such that any viable technique $(\mathbf{A}_\alpha, \mathbf{B}_\alpha)$ consists of at least one process in every sector.

Consider a viable technique $(\mathbf{A}_\alpha, \mathbf{B}_\alpha)$ of the sectoral economy (\mathbf{A}, \mathbf{B}) . If this technique has more than n processes, it follows from a well-known mathematical result (see *e.g.* Gale, 1951,

p. 297) that there always exists a viable technique (A_β, B_β) which has exactly n processes also belonging to technique (A_α, B_α) . Suppose it were possible to find another viable technique (A_γ, B_γ) composed of $n-1$ processes also belonging to technique (A_α, B_α) . In that case the condition set in Definition 8 would be violated. In conclusion, any viable technique extracted from (A, B) is either square and minimal, and therefore all-productive, or contains an all-productive technique. Hence:

Theorem 7. A viable sectoral economy has the super-adjustment property.

The general definition of a sectoral economy does not require the existence of dominant products. This makes the difference: if each process produces a single good or admits a dominant product, the industry or sector to which the process belongs is transparent, since it is determined by the nature of the (main) product. So to say, the name of the sector is engraved on the process itself. Definition 8, on the contrary, starts from a given partition of the set of processes, and breaks the link between a sector and a product. This point is stressed if the commodities are marked by figures $1, 2, \dots, n$ as usual, and the sectors by colours: ‘blue’ ‘white’, etc.... A sectoral economy can then be seen as a ‘rainbow economy’.

Example 2. Consider the system

$$\mathbf{A} = \begin{bmatrix} 27 & 27 & 27 & 27 \\ 27 & 27 & 27 & 27 \\ 27 & 27 & 27 & 27 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 22 & 28 & 24 & 0 \\ 0 & 31 & 39 & 27 \\ 47 & 24 & 19 & 31 \end{bmatrix}$$

(the fact that all processes have the same inputs plays no analytical role and allows to concentrate one’s attention on the output matrix). If the ‘blue’ sector is reduced to process 1, the ‘white’ sector to process 2, and the ‘red’ sector to processes 3 and 4, it can be checked

that a sectoral economy is defined. But it was not clear from the very beginning that this partition is more meaningful than any other.

For theorists, the main feature of Definition 8 lies in its remarkable simplicity. It offers a direct access to the properties of all-productive systems while avoiding matricial calculations. But Definition 8 requires a change in the habits of thought: instead of looking exclusively at the individual processes, it proceeds from the properties of ‘bunches’ of processes. The approach is global, not local.

4.3. A general result

Theorem 7 showed the implication: “sectoral economy \Rightarrow super-adjustment”. The inverse does not hold, however, as can be seen from a numerical example.

Example 3. Consider an economy with $n = 3$ commodities and $m = 5$ processes, characterized by the following data:

$$\mathbf{A} = \begin{bmatrix} 19 & 0 & 0 & 5 & 6 \\ 0 & 20 & 0 & 0 & 24 \\ 0 & 0 & 3 & 4 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 13 & 6 & 0 & 0 \\ 13 & 0 & 7 & 10 & 0 \\ 6 & 7 & 0 & 0 & 10 \end{bmatrix}$$

Given that the sets of processes $\{1,2,3\}$, $\{1,2,4\}$, and $\{3,4,5\}$ all define viable techniques, it is impossible to partition the set of processes $\{1,2,3,4,5\}$ into 3 sectors such that every viable technique contains at least one process from each sector. Hence the economy is not a sectoral economy. Yet it possesses the super-adjustment property.

The important characteristic for super-adjustment is not so much the existence of n sectors, but the fact that any viable technique must consist of at least n processes. This is the meaning of the following theorem which generalizes Theorem 7:

Theorem 8. The system (\mathbf{A}, \mathbf{B}) has the super-adjustment property if and only if:

- (i) the system is viable;
- (ii) no technique $(\mathbf{A}_\alpha, \mathbf{B}_\alpha)$ of $n-1$ processes is viable.

5. THE ADJUSTMENT PROPERTY WITH PURE CAPITAL GOODS

The economic interpretation of the adjustment property assumes that all commodities are consumed. Two kinds of commodities are now distinguished: n goods which may be used in production and for which there is a positive final demand ('consumption goods', such as corn), and k goods which are used for the production of other goods but for which there is a zero final demand ('pure capital goods', such as fertilizers or fixed capital). The notion of pure capital goods is common in Sraffian analysis. It is, however, not always adopted by I-O economists, who usually treat the (intermediate) demand for investment goods as if it were a component of final demand. The consumption goods are the first n commodities, the pure capital goods the last k ones. To reflect this distinction, we partition \mathbf{A} and \mathbf{B} as follows:

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}' \\ \mathbf{A}'' \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{B}' \\ \mathbf{B}'' \end{bmatrix} \quad (21)$$

with the $n \times m$ matrices A' and B' referring to the n consumption goods, and the $k \times m$ matrices A'' and B'' to the k pure capital goods. Since by definition there is no final demand for pure capital goods, we restrict our attention to net output vectors of the following type:

$$d = \begin{bmatrix} d' \\ \mathbf{0} \end{bmatrix} \quad (22)$$

where the $n \times 1$ net output vector of consumption goods d' is semipositive.

The previous definitions must be adapted to the new framework. Definitions 1 and 2 become:

Definition 9. Consider a system (A, B) with pure capital goods, and let $C = (B - A)$. The system is said to be *viable* if it is capable of producing a positive net output vector of consumption goods while producing a zero net output of pure capital goods:

$$\exists y \geq \mathbf{0}, \quad C'y > \mathbf{0}, \quad C''y = \mathbf{0} \quad (23)$$

Definition 10. A system with pure capital goods has the *adjustment property* (or is *adjustable*) if it can produce any semipositive net output vector of consumption goods while producing a zero net output of pure capital goods. Formally:

$$\forall d = \begin{bmatrix} d' \\ \mathbf{0} \end{bmatrix} \geq \mathbf{0}, \quad \exists y \geq \mathbf{0}, \quad C'y = d', \quad C''y = \mathbf{0} \quad (24)$$

The definition of minimality given in Definition 9 is unchanged. In the presence of pure capital goods, Theorem 6 no longer holds, as shown by a numerical example:

Example 4. Let there be $n = 1$ final good, $k = 2$ capital goods and $m = 2$ processes, and consider a system such that:

$$\mathbf{C} = \mathbf{B} - \mathbf{A} = \begin{bmatrix} + & + \\ 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

This system is viable (choose the same activity levels for the two processes), minimal (an isolated process cannot produce a zero net product of capital goods), and has the adjustment property (any final demand of type $(+, 0, 0)$ can be met). Yet it is not square, because there are two processes but three commodities.

However, some relationship between the number of processes and that of commodities is expected. The intuition is: the number m of processes represents the number of degrees of freedom in choosing activity levels. Since the adjustment property contemplates n different directions of demand for final goods, one must have $m \geq n$. The $m - n$ remaining degrees of freedom are used to adjust the net product of pure capital goods to zero. To require them being equal to the number k of pure capital goods would however be too restrictive: as illustrated by the last two rows of Example 4, the capital goods might be linked in such a way that when the capital good X disappears from the net product, so does the capital good Y . What matters is the number of independent capital goods, as measured by the rank of matrix \mathbf{C}'' .

This intuitive accounting is formalized in Definition 12, which is the generalization of the notion of squareness in the new framework. First Definition 11 introduces a condition in order to eliminate ‘superfluous’ processes.

Definition 11. A process belonging to a viable system is *superfluous* if it cannot be used for the production of any semipositive net product. Formally, process i is superfluous if the following implication holds:

$$\{y \geq 0, C'y \geq 0, C''y = 0\} \Rightarrow y_i = 0 \quad (25)$$

Example 5. Starting from Example 4, let us add a third process, the net product of which is 1 in the first capital good and 2 in the second capital good. Since the net product of all capital goods cannot be zero if this process is operated, it is a superfluous process.

Definition 12. A system with pure capital goods is *balanced* if the following two conditions are satisfied:

- The number of processes is equal to the number of final goods plus the rank of matrix C'' :

$$m = n + rk(C'') \quad (26)$$

- It admits no superfluous processes.

Theorem 9 is the generalization of Theorem 6 in presence of capital goods:

Theorem 9. If a system with pure capital goods has any two of the following properties:

- (i) it is minimal;
- (ii) it is balanced;
- (iii) it is adjustable;

it has the third one.

6. VERTICAL INTEGRATION

The theory of fixed capital has dealt with a genuine species of pure capital goods identified as ‘machines’. It proposes to proceed in two steps: first eliminate the machines by means of vertical integration, then consider the integrated or reduced economy with final goods only. An example gives a flavour of that approach.

Example 6. Let there be two final goods, iron and corn, and five processes. Process 1 describes the production of iron (good 1), processes 2 to 5 that of corn (good 2). Process 2 produces a pure capital good, called ‘new tractor’ (good 3), jointly with corn. The new tractor enters into process 3 as an input, and reappears, on the output side, as a ‘one-year-old tractor’ (good 4). Similarly, process 4 uses the one-year-old tractor as an input and produces corn and a ‘two-year-old tractor’ (good 5). In the last process the tractor is worn out.

The corresponding matrices are written as:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} b_{11} & 0 & 0 & 0 & 0 \\ 0 & b_{22} & b_{23} & b_{24} & b_{25} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Over the lifetime of a tractor, the net output of the ‘corn industry’, which consists here of processes 2 to 5, is reduced to corn only. Instead of considering a several-year period, let us look at the economy in a steady position for tractors, *i.e.* let us consider a situation in which the number of ‘births’ and ‘deaths’ of machines are equal. The net product is then reduced to iron and corn. Since there is no joint production of these final goods, the adjustment property is expected to hold.

This Section generalizes the approach to more complex cases. The specific features of Example 6, which do not necessarily show up in the general case, are:

- Matrix $C'' = B'' - A''$ has maximal rank, equal to the number of pure capital goods k .
- In order to eliminate the tractors from the net product, it suffices to apply the same activity levels to all processes in the corn industry. The important fact there is that the algebraic combination of columns which translates this elimination in mathematical terms is a *positive* combination. In more general models this will not always be the case.
- The integrated economy is a single-products economy.

In the general case, the successive steps of the procedure are:

i) If the capital goods are not independent, select r independent rows from C'' . Let us note them \bar{C}'' . \bar{C}'' is an $r \times m$ matrix such that $rk(\bar{C}'') = r$. We can disregard the $(k-r)$ other rows of matrix C'' , which correspond to the number of 'dependent' pure capital goods.

ii) Next, let us identify in \bar{C}'' a number of columns with maximal rank r , and rearrange matrix C'' accordingly. More precisely, let us renumber the processes in order to get the following configuration:

$$\bar{C} = \begin{bmatrix} C'_1 & C'_2 \\ \bar{C}''_1 & \bar{C}''_2 \end{bmatrix} \quad (27)$$

with C'_1 an $n \times (m-r)$ matrix, \bar{C}''_2 an $r \times r$ matrix, etc., and $rk(\bar{C}''_2) = r$ (there may be more than one way to choose matrix \bar{C}''_2). Since matrix \bar{C}''_2 is regular, matrix G

$$G = -(\bar{C}''_2)^{-1} \bar{C}''_1 \quad (28)$$

is well defined. When activity levels y_1 apply to the processes of type 1, the capital goods disappear from the net product if, simultaneously, activity levels $y_2 = Gy_1$ apply to the capital

goods of type 2. Matrix \mathbf{G} , which ‘eliminates’ the pure capital goods, is called the *integration matrix*.

iii) With the same activity levels, the net product of final goods amounts to Γy_1 , with

$$\Gamma = \mathbf{C}'_1 + \mathbf{C}'_2 \mathbf{G} . \quad (29)$$

To sum up, all happens as if, after elimination of the capital goods, there existed a set of $m - r$ processes defined by the columns of matrix Γ , these processes representing the production of final goods by means of final goods only. Γ is called the *integrated matrix*.

Theorem 10. Consider a system with pure capital goods which satisfies condition (26). The adjustment property holds if and only if the integrated matrix Γ is regular and

$$\Gamma^{-1} \geq \mathbf{0} \quad (30)$$

$$\mathbf{G} \Gamma^{-1} \geq \mathbf{0} \quad (31)$$

\mathbf{G} and Γ being defined by (28) and (29). In particular, if the integration matrix \mathbf{G} is semipositive, the system has the adjustment property if and only if the integrated matrix Γ has a semipositive inverse.

In the absence of capital goods Γ is nothing but $(\mathbf{B} - \mathbf{A})$, so that Theorem 10 is a generalization of the properties of all-productive systems as given by Theorem 4.

Example 7. Let us return to Example 6. Matrix \mathbf{C}'' has maximal rank ($r = k = 3$). Partition (27) is obtained by separating the first two columns from the last three ones. Hence:

$$\mathbf{C}'_1 = \begin{bmatrix} b_{11} - a_{11} & -a_{12} \\ -a_{21} & b_{22} - a_{22} \end{bmatrix} \quad \mathbf{C}'_2 = \begin{bmatrix} -a_{13} & -a_{14} & -a_{15} \\ b_{23} - a_{23} & b_{24} - a_{24} & b_{25} - a_{25} \end{bmatrix}$$

$$\bar{C}_1'' = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \bar{C}_2'' = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

$$(\bar{C}_2'')^{-1} = \begin{bmatrix} -1 & 0 & 0 \\ -1 & -1 & 0 \\ -1 & -1 & -1 \end{bmatrix} \quad \mathbf{G} = -(\bar{C}_2'')^{-1} \bar{C}_1'' = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{\Gamma} = \mathbf{C}'_1 + \mathbf{C}'_2 \mathbf{G} = \begin{bmatrix} b_{11} - a_{11} & -(a_{12} + a_{13} + a_{14} + a_{15}) \\ -a_{21} & (b_{22} + b_{23} + b_{24} + b_{25}) - (a_{22} + a_{23} + a_{24} + a_{25}) \end{bmatrix}$$

As pointed out, $\mathbf{\Gamma}$ is the matrix describing the reduced economy with two processes:

$$a_{11} \text{ iron} \oplus a_{21} \text{ corn} \rightarrow b_{11} \text{ iron}$$

$$(a_{12} + a_{13} + a_{14} + a_{15}) \text{ iron} \oplus (a_{22} + a_{23} + a_{24} + a_{25}) \text{ corn} \rightarrow (b_{22} + b_{23} + b_{24} + b_{25}) \text{ corn}$$

where the second process represents the integrated corn industry without net product of tractors.

In the present case matrix \mathbf{G} is semipositive and $\mathbf{\Gamma}^{-1}$ has a semipositive inverse (viability being assumed). Theorem 10 applies and the economy described in Example 6 has the adjustment property. The method followed in this Section and Theorem 9 provide a mechanical device to check whether the adjustment property holds or not.

7. A PROGRAMME FOR FURTHER RESEARCH

The economic properties of single-product systems had been explored by Sraffa himself, so that the scope for further research in this field was quite restricted. On the contrary, Sraffa's book left unsolved a number of puzzles concerning the behaviour of joint production systems: it recognized that some economic laws established for single production do not hold in joint

production, but no precise account of these divergences was given. They are probably more significant than initially expected: in several circumstances, Sraffa underestimated the gap and unduly proceeded by analogy. The study of multiple-product systems has been a central analytical objective of the Sraffian economists, with the permanent idea of a comparison with single production. (The idea itself is not so obvious: let us recall that the distinction between single-and multiple-product systems is not even mentioned in many books on general equilibrium, such as Debreu's *Theory of Value*.) An important part of this research is the identification of the types of joint production systems which behave like single-product systems. Sections 3 to 6 constitute an overview of the results obtained in this field.

This field of research, which has mainly been explored by the Sraffians, is clearly of interest for I-O specialists. The aims of these two categories of economists are different, however, so that the transfer of the Sraffian results to the I-O framework is not immediate. For instance, when the Sraffians identify single-production-like properties, it is never with the idea of substituting a simplified representation (\bar{A}, I) for the complete representation (A, B) . Such a replacement is frequently made in I-O analysis, but we have shown that this may sometimes be misleading (*cf.* subsection 3.4). I-O economists know that their procedure is only an approximation, and generally welcome constructive dialogue with theoretical economists on when the approximation is or is not justified.

Consider the example of negative coefficients. Frequently I-O economists find that the input-output matrix \bar{A} which they construct as a proxy for the observed multiple-production system happens to have negative coefficients (*e.g.* Rainer, 1989). Many of them are very reluctant to accept this 'fact'; often they decide that this must be due to measurement errors and the like, and begin to tamper with the data. Pure theorists condemn this practice. In our opinion, it would be preferable to organize a research programme on such questions as: under

which circumstances and for which problems does a joint production system act *as if* the input-output matrix admitted negative coefficients? Which are the best rules to construct that pseudo-input-output matrix?, etc. As a very first answer, let us report the following result: the conclusions concerning all-engaging and all-productive systems (Section 3) still hold when the input-output matrix has negative coefficients.

To sum up, the identification of the cases when multiple-product systems behave like single-products ones, which is the result of Sraffian analyses, is important. But it is only a first step in a more general research programme for which cooperation between theoretical and applied economists is required.

APPENDIX: A NOTE ON THE SOURCES OF THE THEOREMS

Theorem 1 follows from Theorem 6 and Definition 5. Theorems 2, 3 and 4 are from Bidard (1996a). Theorem 5 can be found in Bidard (1991, ch. XI). Theorems 6, 9 and 10 have been proved in Bidard and Erreygers (1998). Theorems 7 and 8 are derived from the results given in Erreygers (1996).

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Endnotes

¹ The (A, B) notation is standard in both the Sraffa and the von Neumann literature; in input-output economics the convention is to represent the inputs by a *use-matrix* U , and the outputs by a *make-matrix* V^T .

² With regard to vectors and matrices the expression ' $\geq \mathbf{0}$ ' indicates semipositivity, and ' $> \mathbf{0}$ ' positivity.

³ Condition (5) also defines q as the natural generalization of a positive Standard commodity as defined by Sraffa (1960), up to a factor of normalization. Such a Standard commodity does not always exist in joint production (Sraffa, 1960, §53), but g -all-engaging systems are special cases of joint production.

⁴ This can always be accomplished as follows: if one 'old' unit of good j costs p_j \$, then define one 'new' unit as equal to $1/p_j$ 'old' units.