The Fundamental Input-Output Identities with Heterogeneous Prices and Imperfect Economic Accounts [1]

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Abstract *The fundamental input-output identities that express the solution for commodity output and price also guarantee that nominal GDP is the same whether measured as the sum of final demand expenditures or as the sum of value added. In the simplified framework in which the identities are usually presented, all relationships hold exactly, and there is one homogenous price for each commodity. However, when constructing a model of a real economy, one usually needs to confront several issues. First, prices may vary across the row. Second, input-output identities often do not hold with available data . Finally, final demand and value added from the input-output table may not be equal to the corresponding figures from the aggregate national accounts.*

The solution to the first problem is simply to use a more complicated input-output identity. Often, the solution to the last two problems is to introduce discrepancy variables that can be added to final demand, unit value added, or aggregate final demand and value added concepts to force the necessary identities to hold. Such discrepancies are then usually held constant for the forecast interval of the model. This paper first discusses how non-homogenous prices are handled in several Inforum models. Next, we investigate the conditions under which nominal GDP as measured by total final demand or total value added is still equivalent, in the presence of final demand and value added discrepancies. Finally, several other types of discrepancies are discussed, that arise from inconsistent input-output and national accounts data.

1. Introduction

The real world is never as simple as the textbook examples, and real world interindustry models must deal with several problems not often discussed in the input-output textbooks. One of these is the presence of non-homogenous prices, discussed in the first section. There is another large set of problems deriving from inconsistencies in data from various sources. The statisticians in our federal government are of the highest caliber, and produce high quality national accounts, input-output accounts, employment and price data. Still, the needs of interindustry model builders are just a small voice in the cacophony of competing demands on their attention. Consistent and integrated time series of input-output based national accounts are still a long way off, at least in the U.S. The last three sections of this paper discuss various kind of discrepancies that have been adopted to deal with these problems, and where possible, I discuss alternatives to using discrepancies.

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2. Heterogeneous Prices

The fundamental input-output identities are usually stated as

(2.1)
$$q = Aq + f$$
$$p' = p'A + v'$$

where:

- q is the vector of constant price commodity output
- p is the vector of producer prices by commodity
- A is the direct requirements matrix, with a_{ij} defined as the ratio of constant price intermediate input from sector *i* used to produce output of sector *j*, divided by the constant price output of sector *j*
- f is the vector of constant price final demand, with constant price imports included as a negative
- v is the vector of unit value added, i.e. $v_i = V_i / q_i$, where V is the vector of value added

These fundamental equations are beautiful in their simplicity and grand in concept. Together, they imply what may well be called the fundamental theorem of input-output analysis[2]

(2.2)
$$v'q = p'f$$

which should hold at any vector of output *q* and any vector of prices *p'* as long as equations 2.1 hold. Equation 2.2 is just another way of stating the familiar identity of income and expenditure in national income accounting. Since current price output is the same, whether measured as the sum of sales to intermediate and final demand or as the sum of intermediate costs and value added, the netting out of intermediate flows results in the equality in the sums of the cells in the final demand and value added quadrants of the input-output tableau.

The system described above assumes the same price for imports as for domestically produced goods and services. However, separate price data are usually available for imports, and they may have movements quite different from domestic prices. The system in 2.1 can be modified as follows

(2.3)
$$q = A_m q + A_d q + f_m + f_d - m$$
$$p'_d = p'_d A_d + p'_m A_m + v'$$

where:

A_m is the *import requirements matrix*, with a_{ij}^m equal to the ratio of imports of sector *i* used as

[2] Almon (1997) on page 10 states and names this theorem, sketching a short proof in a footnote. See also Stone (1961), p. 96. Surprisingly, the topic is not touched in Miller and Blair (1987).

intermediate inputs in the production of sector *j*

A_d is the *domestic requirements matrix*, with a_{ij}^d equal to the ratio of domestic intermediate requirements of sector *i* used in the production of sector *j*

f_m is final demand supplied by imports

f_d is final demand supplied by domestic production

m is imports

p_d is the price of domestically produced output

p_m is the import price

and

(2.4) $$A_m q + f_m - m = 0$$

If we multiply the domestic terms of the first equation in (2.3) by p_d' and the import terms by p_m', the resulting equation still holds, since the import terms still sum to zero. The second equation in (2.3) can be postmultiplied by *q* yielding

(2.5) $$\begin{aligned} p_d'q &= p_m'A_m q + p_d'A_d q + p_m' f_m + p_d' f_d - p_m' m \\ p_d'q &= p_m'A_m q + p_d'A_d q + v'q \end{aligned}$$

Combining the equations and rearranging terms yields another version of the fundamental theorem:

(2.6) $$v'q = p_m' f_m + p_d' f_d - p_m' m$$

This result simply assures us that the measure of GDP as the sum of value add or the sum of final demand is still identical, even with the different valuation of imports.

In practice, separate measures of A_m and f_m are often not available, as is the case with the U.S. input-output accounts. In this case, one can use a *weighted price* p_w which is a weighted average of the domestic and import prices. The derivation of the appropriate weighted price is easier to see by examining a single sector *i*. (Matrix and vector subscripts are written as superscripts, for easier reading.) Rearranging and rewriting the first equation in 2.5 gives[3]

(2.7) $$p_i^w \left[\sum_j (a_{ij}^m + a_{ij}^d) q_j + f_i^m + f_i^d \right] = p_i^d q_i + p_i^m m_i$$

or

[3] In the U.S. model, exports are assumed to be produced entirely domestically. The computation of the weighted price in the U.S. case is only slightly more complicated than given in 2.7 to 2.10.

(2.8) $$p_i^w = \frac{p_i^d q_i + p_i^m m_i}{\sum_j a_{ij} q_j + f_i}$$

where:

$$a_{ij} = a_{ij}^m + a_{ij}^d$$
$$f_i = f_i^m + f_i^d$$

Since the denominator in 2.8 is just $q_i + m_i$ the weighted price can then be expressed as

(2.9) $$p_i^w = p_i^d \frac{q_i}{q_i + m_i} + p_i^m \frac{m_i}{q_i + m_i}$$

The domestic price equation is then written as

(2.10) $$p_d' = p_w' A + v'$$

There are other cases in which a component of final demand may be priced differently from domestic output. For example, in some systems of accounts, separate prices are derived for defense purchases, which are quite different in kind from commercial products in the same industry classification. In Japan, a separate export price is available, reflecting the fact that Japanese exporters price discriminate, and reduce export prices to increase world market share, raising domestic prices to compensate. How should the system be modified to account for a situation like this? We'll examine the question of a different export price, ignoring the issue of import prices for the moment. Assume a price p_o for 'other' goods (all columns except exports, in this case). Then, write the system in both constant and current prices as

(2.11) $$\begin{aligned} q &= Aq + f + x \\ p_d' q &= p_o' Aq + p_o' f + p_x' x \end{aligned}$$

The solution for p_o can then be written as

(2.12) $$p_i^o = \frac{p_i^d q_i - p_i^x x_i}{q_i - x_i}$$

where again the superscripts are used to indicate what were subscripts for the vectors. The domestic price equation in this case is

(2.13) $$p_d' = p_o' A + v$$

As in the other cases, there is a nominal GDP identity which can be easily derived.

(2.14) $$v' q = p_o' f + p_x' x$$

In none of the cases 2.3, 2.10 or 2.13 is a simple closed form solution for p_d' available as in the simplest well-known case:

(2.15) $$p_d' = (I - A)^{-1} v$$

However, the Seidel algorithm can be adopted to handle any of the above cases. The Seidel algorithm is an iterative solution. We make an initial guess of prices, and then improve upon it in each iteration. For example, if we denote the *k*'th iteration estimate of p_i as p_i^k then we can estimate the (k+1)st approximation by the formula

(2.16) $$p_j^{k+1} = \frac{\sum_{i<j} a_{ij} p_i^{k+1} + \sum_{i>j} a_{ij} p_i^k}{1 - a_{ii}}$$

In the case of 2.10, the formula is easily modified to

(2.17) $$p_j^{d,k+1} = \frac{\sum_{i<j} a_{ij} p_i^{w,k+1} + \sum_{i>j} a_{ij} p_i^{w,k}}{1 - a_{ii}}$$

where the notation $p_i^{w,k}$ refers to the kth iteration of the weighted price for sector *i*. Within the loop iteration, each successive estimate of $p_i^{w,k}$ can be calculated according to 2.9.

In summary, the presence of nonhomogeneous prices does not pose a problem for the consistent calculation of outputs and prices. The use of the Seidel iterative algorithm allows for the adoption of complex price identities quite easily. A rigorous check of the computations is to ensure that current price GDP is equal whether calculated as the sum of value added or the sum of current price final demands.

3. Final Demand and Value Added Discrepancies[4]

A typical problem that the model builder faces when constructing an interindustry model is that there may be no consistent data available for recent years. At the time of this writing, output and prices data are available for the U.S. until 1999, but the latest I-O table is an annual estimate for 1997. The latest benchmark I-O table is for 1992. The estimates of output, final demand, value added and prices are all derived from different sources, and have not been forced into consistency. Instead, in the forecasting model, we generally calculate discrepancies for final demand and value added, and then project those discrepancies into the future.

We have

(3.1)
$$q \neq Aq + f$$
$$p' \neq p'A + v'$$

The system is forced into equality through the introduction of discrepancies ε and δ:

(3.2)
$$q = Aq + f + \varepsilon$$
$$p' = p'A + v' + \delta$$

The application of these discrepancies in the model solution can be done in several ways. One is to calculate ε and δ in the last year of output and price data, and then apply them as an additive constant during the Seidel algorithm in forecast years. This has the advantage of being simple, and smoothing the transition from history to forecast. However, the GNP identity is no longer ensured, for we have

(3.3)
$$v'q + \delta' q = p'f + p'\varepsilon$$

If we still require that $v'q = p'f$ then we must have

(3.4)
$$\delta' q = p'\varepsilon$$

This equation basically states that the sum of the value added discrepancies must equal the sum of the final demand discrepancies in current prices. If this is not the case, then the forecast of current price GDP as calculated on the value added side will diverge from the forecast of GDP on the product side.[5] One tempting

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4 Leontief (1953,1966) dealt with this issue in the context of the construction of input-output tables, where he discussed the solution of introducing an 'undistributed column' and 'undistributed row'. For example, see the table on pp. 16-19 in Leontief (1966). This column and row were clearly treated as production sectors, and to obtain a balanced table, total undistributed had to sum to the same total by row or by column. Salkin (1981) discusses statistical problems in the compilation of output time series and input-output accounts. McCarthy (1991) is the only mention I have found of the treatment of discrepancies in a forecasting framework.

5 How fast it will diverge depends upon the differences in average growth rates of final demands and prices. If prices are growing, on average, 2 percent faster than final demands, then the final demand discrepancy total will grow 2 percent faster than the value added discrepancy total.

solution to this problem is to define a new sector *D*, called 'discrepancy', with zeroes in the row and column of the A-matrix. We allocate the difference between $\delta' q$ and $p' \varepsilon$ to that sector, defining

$$\Delta = \delta' q - p' \varepsilon$$

(3.5) $$p_D = v_D + \delta_D$$

$$q_D = f_D + \varepsilon_D$$

where: $\delta_D = \Delta$, $v_D = 0$, $f_D = 1$, and $\varepsilon_D = 0$. Then $p_D = \Delta$ and $q_D = 1$ and equation 3.4 holds for the expanded industry set. This implies that $v' q = p' f$ for the expanded industry set.

However, this is cheating, as we have not really brought value added GDP into equality with product side nominal GDP for the true industries (excluding the discrepancy industry), but have merely added Δ to product side GDP. This can be seen by observing that for industry *D*, $p_D f_D = \Delta$, and $v_D q_D = 0$.

Perhaps a better way is to adopt some sort of scaling procedure for the discrepancies. In other words, after an initial calculation of *q* and *p*, form the ratio

(3.6) $$R = \frac{p' \varepsilon}{\delta' q}$$

and then use a revised estimate of δ

(3.7) $$\hat{\delta}^{k+1} = R \hat{\delta}^{k}$$

in the next calculation of *p*. Intuitively, it seems that this process should converge, and result in estimates of current price GDP that are equivalent on the value added and product side. However, we have not yet tried this.

This section has demonstrated that the technique of forming final demand and value added discrepancies and applying them in a forecast as a constant term adjustment will likely lead to measures of current price GDP that diverge when measured as the sum of value added or the sum of nominal final demand. The adoption of a scaling factor for the price discrepancies would be one solution to this problem. Another solution would be to do away with the discrepancies altogether. In order to do this, we would need to derive a completely consistent input-output tableau for each year, or at least the most recent year of output data.

4. Final Demand Category Price Discrepancies

In many countries, final demand bridge matrices are produced as part of the set of benchmark input-output accounts. For example, in the U.S. accounts, there is a consumption bridge matrix, which we shall call *D*, which relates personal consumption expenditures by category from the national accounts to input-output commodity. Assume there are *N* input-output sectors, and *M* personal consumption categories. Then *D* is of dimension N by M. The columns of the bridge matrix sum to unity, and the bridge is used both to translate personal consumption expenditures by category to personal consumption final demand by input-output commodity as well as to calculate the price deflator for consumption goods by category from the input-output domestic price deflator.

(4.1)
$$C = DJ$$
$$p'_J = p'_w D$$

where:

J is an M by 1 vector of personal consumption by category
p'_J is an M by 1 vector of prices for J
C is an N by 1 vector of personal consumption by input-output commodity
p'_w is the N by 1 vector of weighted prices for input-output commodities

It is usually the case that there is newer data for consumption by category from the national accounts than the input-output data or the bridge. For example, the detailed consumption data from the latest unpublished NIPA is through the year 2000. The most recent consumption bridge was published in 1992, but there are annual input-output accounts available for 1996 and 1997 which provide a personal consumption vector which can be used as a row control for an updated bridge matrix.

With a consumption bridge matrix updated to 1997, we can estimate updated vectors of consumption by input-output category using the first equation in 4.1. The problem arises with the price computation. Since the NIPA data provide both current and constant price personal consumption expenditures by detailed category, estimates of p_J can be derived from 1998 to 2000. However, we have consistently found that, in the last year of data for the domestic input-output commodity deflator, we have

(4.2)
$$p'_J \neq p'_w D$$

The inequality may arise from two sources of error: either the bridge D is not appropriate for that year, or the commodity price vector p_w is not exact for the vector of personal consumption by commodity.

The left and right hand side of 4.2 can be forced into equality by adding a discrepancy term, as we did in the input-output computation. This smooths the transition from the historical series of the consumption deflator to the forecast. This is very important, because these consumption prices are used in the consumption equation system. The consumption prices do not only serve the function of stimulating or depressing demand for individual goods and services, but also of determining total nominal consumption expenditures, which provide an aggregate control total for the system.

However, the use of the price discrepancy means that nominal personal consumption expenditures by category no longer sum to the same total as nominal personal consumption expenditures by input-output commodity. If the total nominal personal consumption by category is used to construct nominal GDP, that nominal GDP will be different from the sum of value added. This is a problem we still have in the *IdLift* model of the U.S.

There are several ways to deal with this problem. Our approach has been to use only the personal consumption vector by commodity in current prices when calculating nominal GDP on the product side, and to ignore nominal consumption by category. Another approach would be to create a new consumption bridge for each year by row and column scaling the nominal bridge matrix to nominal controls. This has been the approach we have followed with the equipment investment bridge matrix. However, in the case of the consumption bridge, there may be no way to scale the bridge that would satisfy both current and constant dollar controls.[6]

This remains a thorny problem, and we are determined to find a solution.

5. Discrepancies Between the National Accounts and the Input-Output Accounts [7]

In the U.S. we have the problem that there is a difference between total personal consumption or GDP as measured in the input-output accounts, and as measured in the national accounts.[8] Although the input-output accounts are usually consistent with the national accounts when they are first derived, the national accounts are subsequently revised, and no corresponding revisions are made to the input-output accounts. With the adoption of more timely annual input-output accounts in the U.S., this problem is not as severe as it once was, but with the next benchmark tables due to be produced on the NAICS basis[9], there will be some delay before their publication, and by that time the NIPA will probably have been revised again.

GDP in the 1987 benchmark input-output table is $4572.0 billion, compared to the current NIPA value of $4742.5 billion. In the 1992 benchmark table, GDP is $6233.9 billion, compared to $6318.9 billion in the NIPA. The annual input-output tables for 1996 and 1997 both agree with NIPA for GDP. One significant change in the NIPA since the publication of the 1992 benchmark table has been the treatment of computer software as investment. However, changes in the accounting for government GDP have also been made during this time. The U.S. has adopted the SNA standard in the treatment of government investment and capital consumption, and there have been significant differences in the treatment of government retirement accounts.

The Bureau of Economic Analysis has not published a revised 1992 benchmark table consistent with the national accounts. However, Inforum has revised the 1992 direct requirements table and investment bridge to adopt the treatment of software as equipment investment. For this work we have drawn upon information in the 1996 annual table.[10] The incorporation of government capital consumption requires only the introduction of another column in each of the government final demand bridges, with an entry only in the "government industry" row. These changes bring us closer to NIPA GDP. In the *IdLift* model, the 1987 value for GDP is now $4755.8 billion, which is somewhat higher than the NIPA GDP of $4742.5 billion.

Since it is desirable for the model to report a value of nominal GDP that is consistent with published national accounts data, we can create a discrepancy for each major component of GDP in the input-output accounts that measures the difference between the total industry data and the corresponding national accounts item.

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[6] In the case of equipment investment, we do not use the national accounts deflators for investment by purchasing industry, but derive our own, based on the commodity output deflators and the investment bridge.

[7] Lawson (1997) presents the benchmark table for 1992. Okubo, et. al. (2000) present the 1996 annual I-O table.

[8] See Meade (1997) pp. 72-73 for a discussion of this problem in the context of the Japanese I-O and SNA.

[9] NAICS is the North American Industry Classification System, described at http://www.census.gov/naics. It is already the system used to publish the 1997 Economic Census, and is due to be standard by 2002.

[10] However, the annual input-output tables provide no detailed equipment investment bridge.

In the last year of national accounts data, these discrepancies can be held constant and added to the industry totals in the forecast period.

Alternatively, we could revise the input-output data so that it sums exactly to the national accounts. The first step is to revise the final demand vectors to agree to the national accounts, either by scaling, or selective changes in individual entries. In the next step we have two choices. One is to revise the output data as the sum of intermediate and the revised final demand, and then recalculate certain components of value added. The other is to preserve the output data, and revise the intermediate flows. In the latter case, value added must also be revised.

With the availability of timely annual input-output tables from the Bureau of Economic Analysis, our task is much easier. These tables already agree with the NIPA, and it remains for us to derive a detailed consumption bridge and investment bridge, as well as determine how the input-output version of value added should be reconciled to the value added in the national accounts.

6. Conclusions

In this short paper I have touched upon several issues confronting the interindustry model builder whose goal is to produce realistic forecasts of an economy, using available published data. I've shown that the various cases of heterogeneous prices present no problems to a consistent model solution. On the other hand, some commonly used technqiues of introducing discrepancies to smooth the link between historical and forecast data do introduce inconsistencies in the two alternative measurements of current price GDP.

In another context, it was shown that such inconsistencies imply problems for consistent SNA balances.[11] For the same reasons, they lead to inconsistent forecasts of the main NIPA tables in the U.S. This problem must be solved before such consistent forecasts can be produced.

[11] Meade (1997), in the discussion of the Japanese SNA.

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