# Interindustrial Effects of Labor Productivity: an Empirical Study for China[1]

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Abstract

Labor productivity is a key element in analyzing an economy. This paper presents an approach to assess the interindustrial effects of labor productivity, based on Leontief's input-output framework. The innovation is that through the calculation of the cost effect and output effect coefficients, the driving and driven sectors of labor productivity can be identified, then the selection of key industries to promote overall labor productivity is available. Attached empirical study for China economy provides a sound result of new approach.

**Key Words**: Input-Output Analysis, Labor Productivity, Economic Growth.

**JEL Code**: C67, J24, O53

Productivity, evoking the relationship between the quantity of goods and services produced, or output, and the quantity of labor, capital, land, energy, and other resources that produced it, the input, is a key element in analyzing an economy, for it illustrates both the efficiency of industry and the wealth-generating capability of the economy.

Because labor is the most easily measured input, labor productivity, output to labor input, presents a tool not only for analyzing productivity, but also for examining labor costs, real income, and employment trends, the important indices to most economies. But up to now there is little knowledge about the interindustrial relations of labor productivity and their effects on overall economy. [1] This paper is one of my continuous efforts to answer questions issued: is there any sector which has substantial effects on other sectors' labor productivity growth? which index can be used to identify it if such sector exists?

Through the calculation of the cost effect and output effect coefficients derived from Leontief's input-output framework, this paper first presents an approach to identify the driving and driven sectors of labor productivity, and select the key industries to promote overall labor productivity. Then an empirical study for China economy provides the essential context for approach evaluation. Finally, summary and directions are given for further research.

## 1. Methodology[2]

Since labor productivity entails the dependence between output and labor input, it is useful to investigate its interindustrial effects based on Leontief's Input-Output framework.

1.1 Consider the following classical input-output model:

| | | Production Sectors | Final Demands | Total Output |
|---|---|---|---|---|
| Production Sectors | | $\{X_{ij}\}_{nxn}$ | $Y$ | $X$ |
| **Primary** | Depreciation | $D^T$ | | |
| | Wages & Salaries | $V^T$ | | |
| **Input** | Profits & Taxes | $M^T$ | | |

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[2] Please see detailed model building in Appendix or [Pan (1992)].

| | | | |
|---|---|---|---|
| Total Input | $X^T$ | | |

Let $a_{ij} = X_{ij} / X_j$ the direct input coefficient,
$d_j = D_j / X_j$ the direct depreciation coefficient,
$v_j = V_j / X_j$ the direct wages & salaries coefficient,
$m_j = M_j / X_j$ the direct profits & taxes coefficient,
and $s_{ij} = X_{ij} / X_i$ the direct distribution coefficient.

Given a basic assumption of the narrow sense of labor productivity growth as:

$$\begin{cases} P_j = X_j / L_j \\ \Delta V_j = 0 \quad and \quad \Delta L_j = 0 \end{cases}$$

here $P_j$: labor productivity of sector j,
$L_j$: number of staff and workers of sector j,
which means output growth is not supported by labor input.[1]

Define $H = [I - (\hat{D} + \hat{M})]^{-1} S^T \qquad s_j = 1 - (d_j + m_j) = \sum_{i=1}^{n} a_{ij} + v_j$

here $\hat{D}$=diag{$d_1,\ldots, d_n$}, $\hat{M}$=diag{$m_1,\ldots,m_n$}

then
$$H = \begin{pmatrix} \frac{1}{s_1} & & \\ & \ddots & \\ & & \frac{1}{s_n} \end{pmatrix} \begin{pmatrix} s_{11} & \cdots & s_{n1} \\ \vdots & \ddots & \vdots \\ s_{1n} & \cdots & s_{nn} \end{pmatrix} = \begin{pmatrix} \frac{s_{11}}{s_1} & \cdots & \frac{s_{n1}}{s_1} \\ \vdots & \ddots & \vdots \\ \frac{s_{1n}}{s_n} & \cdots & \frac{s_{nn}}{s_n} \end{pmatrix}$$

Consider the sum of its row element:

$$c_j = \frac{s_{1j}}{s_j} + \cdots + \frac{s_{nj}}{s_j} = \frac{\sum_{i=1}^{n} s_{ij}}{v_j (1 + \sum_{i=1}^{n} a_{ij} / v_j)}$$

If the value of $c_j$ is great, it means that sector j has high backward linkage and /or the labor input in its direct cost structure is low, which states this sector has few potential of labor productivity growth and /or has little effect on other sectors' labor productivity growth. Labor productivity growth of this sector mainly depends on labor productivity growth of other sectors, so this kind of sector can be called the **driven sector of labor productivity**.

On the other hand, if the value of $c_j$ is low, it means that sector j has little backward linkage and /or the labor input in its direct cost structure is high, which states this sector has plentiful potential of labor productivity growth and /or has great effect on other sectors' labor productivity growth. Labor productivity growth of this sector will promote labor productivity of other sectors, so this kind of sector can be called the **driving sector of labor productivity**.

Therefore we define $c_j$ the **cost effect coefficient** of labor productivity.

1.2 Let $d_j =$ output growth rate of sector j

$r_j =$ labor productivity growth rate of sector j

i.e.
$$\begin{cases} \Delta X_j = d_j X_j \\ \Delta P_j = r_j P_j = r_j X_j / L_j \end{cases}$$

Consider the effect of labor productivity growth of sector i on other sectors and whole economy, we can get:

[1] L also can be defined as input of labor time. This is not significant because L will be reduced in following model building.

$$\begin{cases} d_1 = \dfrac{s_{i1}}{s_1} r_i \\ \cdots \\ d_n = \dfrac{s_{in}}{s_n} r_i \end{cases}$$

which reflects the influence on other sectors.

Let $w_j = X_j / \sum_{j=1}^{n} X_j$ (weighted coefficient)

then the growth rate of total output:

$$d = \sum_{j=1}^{n} w_j d_j = \sum_{j=1}^{n} w_j \frac{s_{ij}}{s_j} r_i = r_i \sum_{j=1}^{n} w_j \frac{s_{ij}}{s_j}$$

Define $$e_i = \sum_{j=1}^{n} w_j \frac{s_{ij}}{s_j}$$

because $e_i$ reflects the level of influence on whole economy when each $r_i (i = 1, \ldots, n)$ shares the same value, it can be called the **output effect coefficient** of labor productivity.

Now we can clearly see what the economic sense of matrix *H* is:

its element $h_{ij} = s_{ji} / s_i$ means the linked coefficient of labor productivity growth of sector j to output growth of sector i, therefore *H* can be defined the **structural influence matrix** of labor productivity.

1.3 According to above analysis, we should choose the sectors whose cost effect coefficient is small and output effect coefficient is big as the key sectors to promote labor productivity.

## 2. Empirical Study for China

China has already achieved nearly two decades economic booming since her Open-up and reform, and issued various comments on her growth and performance, particularly the productivity.[2] However, according to contemporary international standard, China's productivity still rather lower compared to the developed countries and most new industrializing countries and regions (see table 1).

Table 1 1996 International Overall Productivity Comparison

Unit: GDP per person employed, in US$

| State | Productivity | Rank | Ratio to China |
|---|---|---|---|
| Luxembourg | 80, 328 | 1 | 71. 8 |
| Japan | 70, 919 | 5 | 63. 4 |
| USA | 59, 792 | 11 | 53. 5 |
| Hong Kong | 51, 569 | 15 | 46. 1 |
| Singapore | 50, 050 | 17 | 44. 8 |
| Taiwan | 30, 114 | 25 | 26. 9 |
| Korea | 23, 391 | 29 | 20. 9 |
| Malaysia | 12, 126 | 33 | 10. 8 |
| Brazil | 7, 338 | 40 | 6. 6 |
| Russia | 6, 685 | 41 | 6. 0 |
| Thailand | 4, 418 | 42 | 4. 0 |
| Philippines | 3, 072 | 43 | 2. 7 |
| Indonesia | 2, 411 | 44 | 2. 2 |
| China | 1, 118 | 45 | |

Data Source:World Competitiveness On-line,Ranking as of May 16,1997

Concerning 1.2 billion of population of China in which over 0.8 billion live in rural area, in table 1 we use GDP per person employed, not GDP per capita, to denote her overall productivity for relatively accurate comparison. These too huge productivity gaps indicate that China still need very hard efforts in next decades to catch up the economic efficiency, nonetheless to shorten domestic productivity gap to keep her social order in this uneven world.

Then it is valuable to decompose overall productivity to industrial level, by using above approach, to look detailed scenarios (see Table 2).

Table 2 Key Sector Identification

| Sector | No. | Forward linkage | Backward linkage | Cost effect coefficient | Norm. C. | Output effect coefficient | Norm. E. |
|---|---|---|---|---|---|---|---|
| Agriculture | 01 | 0.699 | 0.686 | 0.987 | 1.128 | 0.044 | 1.127 |
| Coal mining | 02 | 1.296 | 0.934 | 0.335 | 0.383 | 0.035 | 0.887 |
| Crude petro. & N. gas prod. | 03 | 1.705 | 0.755 | 0.382 | 0.437 | 0.030 | 0.769 |
| **Metal ore mining** | 04 | 2.105 | 0.967 | 0.266 | 0.304 | 0.062 | 1.600 |
| **Other mining** | 05 | 1.200 | 0.923 | 0.303 | 0.347 | 0.064 | 1.644 |
| Food manufacturing | 06 | 0.597 | 0.974 | 0.943 | 1.078 | 0.031 | 0.807 |
| Manufacture of textiles | 07 | 0.939 | 1.182 | 0.958 | 1.095 | 0.037 | 0.953 |
| Manu. wear.app.,leather, etc. | 08 | 0.446 | 1.206 | 0.461 | 0.527 | 0.006 | 0.154 |
| **Sawmills & manu. furniture** | 09 | 0.917 | 1.147 | 0.378 | 0.432 | 0.044 | 1.143 |
| Manu. paper, culture & edu. | 10 | 0.933 | 1.109 | 0.652 | 0.745 | 0.034 | 0.862 |
| Elec., steam & hot water P&S | 11 | 1.379 | 0.857 | 1.029 | 1.176 | 0.050 | 1.291 |
| Petroleum refineries | 12 | 1.369 | 0.949 | 1.073 | 1.226 | 0.058 | 1.488 |
| *Coking, manu. gas & coal pro* | 13 | 1.099 | 1.098 | 0.179 | 0.204 | 0.035 | 0.895 |
| *Chemical industries* | 14 | 1.237 | 1.091 | 1.990 | 2.275 | 0.072 | 1.838 |
| Manu. building mat. & non-me | 15 | 0.979 | 1.021 | 1.402 | 1.602 | 0.063 | 1.621 |
| *Primary metal manu.* | 16 | 1.535 | 1.094 | 2.739 | 3.131 | 0.063 | 1.608 |
| **Manu. of metal products** | 17 | 0.950 | 1.175 | 0.660 | 0.755 | 0.049 | 1.263 |
| Manu. of machinery | 18 | 0.942 | 1.133 | 1.571 | 1.797 | 0.037 | 0.939 |
| *Manu. transport equipment* | 19 | 1.011 | 1.164 | 0.707 | 0.808 | 0.033 | 0.836 |
| Manu. electric mach.& instru. | 20 | 0.926 | 1.165 | 0.724 | 0.828 | 0.034 | 0.869 |
| Manu. electronic & comm. equ | 21 | 0.819 | 1.187 | 0.669 | 0.765 | 0.016 | 0.411 |
| ***Manu. instru., meters, etc.*** | 22 | 1.293 | 1.070 | 0.201 | 0.230 | 0.040 | 1.032 |
| Maint. & repair of M&E | 23 | 0.570 | 1.120 | 0.168 | 0.192 | 0.014 | 0.363 |
| ***Industries not classified*** | 24 | 1.338 | 1.163 | 0.437 | 0.499 | 0.063 | 1.628 |
| Construction | 25 | 0.378 | 1.100 | 2.225 | 2.543 | 0.003 | 0.074 |
| Freight trans.& communicatio | 26 | 1.150 | 0.812 | 1.151 | 1.316 | 0.069 | 1.783 |
| Commerce | 27 | 1.033 | 0.870 | 2.534 | 2.897 | 0.045 | 1.161 |
| Restaurants | 28 | 0.346 | 0.898 | 0.200 | 0.229 | 0.000 | 0.000 |
| Passenger transport | 29 | 0.683 | 0.834 | 0.415 | 0.474 | 0.028 | 0.720 |
| Public utilities &ser. to hou. | 30 | 0.902 | 0.781 | 0.894 | 1.023 | 0.041 | 1.054 |
| Cul. edu. health&sci. re. ins. | 31 | 0.539 | 0.838 | 0.711 | 0.813 | 0.016 | 0.423 |
| **Finance & insurance** | 32 | 1.339 | 0.821 | 0.798 | 0.913 | 0.068 | 1.755 |
| Public administration | 33 | 0.346 | 0.879 | 0.723 | 0.827 | 0.000 | 0.000 |

* Forward linkage, calculated from total distribution coefficient, and backward linkage, calculated from total demand coefficient, are both taken Hirschman normalization treatment;

Norm. C. and Norm. E. are the value of Hirschman normalization of Cost effect coefficient and Output effect coefficient, respectively;

Calculation based on China Input-Output Table 1992 (33 sectors, in value unit).

We list traditional linkage analysis results (backward linkage and forward linkage) in Table 2 for approach comparison. According to Hirschman standard, we should select the industries of No. 13, 14, 16, 19, 22 and 24 as key sectors (*Italic* marked), all the manufacturing industries. But by above interindustrial labor productivity approach, we should select the industries of No. 4, 5, 9, 17, 22, 24 and 32 as key sectors (**Bold** marked), in which No. 4 (Metal ore mining) and 5 (Other mining) belong to basic industry, 9 (Sawmill & manu. Furniture), 17 (Metal products), 22 (Instruments & meters) and 24 (Industries not classified) belong to manufacturing industry, and 32 (Finance & insurance) belong to service industry. Since it is mining & quarrying, and finance & insurance (or service), not manufacturing, the "bottle neck" of Chinese economy today and foreseeable future, we'd say that this empirical study well prove the significance of interindustrial labor productivity approach.

Now let's further investigate above identified key sectors to study the relationships between labor productivity (see Table 3), employment (see Table 4) and welfare (see Table 5):

Table 3 Growth of Labor Productivity of China*

Unit: RMB Yuan /person year

| Identified Sectors | 1987 | 1992 | 1996 | R(92/87) | R(96/92) |
|---|---|---|---|---|---|
| Ferrous metals mining and dressing | 6, 071 | 17, 021 | 15, 937 | 2. 80 | 0. 94 |
| Nonferrous metals mining and dressing | 6, 665 | 18, 071 | 16, 222 | 2. 71 | 0. 90 |
| Nonmetal minerals mining and dressing | 4, 766 | 14, 758 | 13, 752 | 3. 10 | 0. 93 |
| Timber Processing, etc. | 7, 253 | 18, 287 | 14, 540 | 2. 52 | 0. 80 |
| Furniture manufacturing | 8, 702 | 19, 910 | 16, 004 | 2. 29 | 0. 80 |
| Coking and coal gas production | 11, 369 | 29, 230 | | 2. 57 | |
| Metal products | 11, 028 | 28, 074 | 17, 886 | 2. 55 | 0. 64 |
| Instruments | 10, 867 | 24, 639 | 15, 341 | 2. 27 | 0. 62 |
| **Total Industry** | 13, 961 | 34, 338 | 22, 018 | 2. 46 | 0. 64 |
| Fi nance | 30, 901 | 76, 824 | | 2. 49 | |

Data Sources: China Statistical Yearbook 1988, 1993, 1997

* R(92/87)=LP(92)/LP(87), R(96/92)=LP(96)/LP(92); Blank: data unavaliable

The identified sectors' labor productivity had increased much quicker from 1987 to 1992 and decreased slower from 1992 to 1996 compared to other sectors. The decline of China's national labor productivity in 1992-1996 mainly contributes to the booming of township and village enterprises (TVEs) which generally adopt traditional or outdated technologies and equipment because of capital shortage, though TVEs employ millions of rural labors. This empirical result further confirms the function of interindustrial effects of labor productivity.

Table 4 Employment of China*

Unit: 10 thousand

| Identified Sectors | Number of Staff and Workers | | | Ratios (%) | | |
|---|---|---|---|---|---|---|
| Year | 1987 | 1992 | 1996 | 1987 | 1992 | 1996 |
| Ferrous metals mining and dressing | 20. 9 | 24 | 21 | 0. 35 | 0. 36 | 0. 33 |
| Nonferrous metals mining and dressing | 57. 1 | 68 | 60 | 0. 96 | 1. 03 | 0. 93 |
| Nonmetal minerals mining and dressing | | | 58 | | | 0. 90 |
| Timber Processing, etc. | 41. 7 | 73 | 72 | 0. 70 | 1. 10 | 1. 12 |
| Furniture manufacturing | 39. 1 | 40 | 31 | 0. 65 | 0. 60 | 0. 48 |
| Coking and coal gas production | 18 | 29 | | 0. 30 | 0. 44 | |
| Metal products | 152. 3 | 184 | 181 | 2. 55 | 2. 78 | 2. 81 |
| Instruments | | 72 | 82 | | 1. 09 | 1. 27 |
| **Total Industry** | 5971 | 6621 | 6450 | 45. 19 | 44. 76 | 43. 45 |
| Fi nance | 154 | 223 | 288 | 1. 17 | 1. 51 | 1. 94 |
| National Total | 13214 | 14792 | 14845 | | | |

Data Sources: China Statistical Yearbook 1988, 1993, 1997

* to industries, ratio=number of sector/total industry *100

to total industries and Finance, ratio=number of sector/national total *100

Blank: data unavaliable

Employment ratios of identified sectors keep stable or increasing from 1987 to 1996, while employment of total industry decreasing. Combining the scenarios of table 3 and 4, it is obvious that labor productivity growth of identified sectors is much faster that other industries.

Table 5 Average Wages of Staff and Workers

Unit: RMB Yuan /Year, current price

| Sectors | 1987 | 1992 | 1996 | R(92/87) | R(96/92) |
|---|---|---|---|---|---|
| Mining & Quarrying | 1, 663 | 3, 209 | 6, 482 | 1. 93 | 2. 02 |
| Manufacturing | 1, 418 | 2, 635 | 5, 642 | 1. 86 | 2. 14 |
| Finance & Insurance | 1, 458 | 2, 829 | 8, 406 | 1. 94 | 2. 97 |
| National Total | 1, 459 | 2, 711 | 6, 210 | 1. 86 | 2. 29 |

Data Sources: China Statistical Yearbook 1988, 1993, 1997

Since real economic growth can only be sustained by fast increase of labor productivity and slow increase of wages or incomes, combining the scenarios of table 3 and 5, we have to say in 1987-1992 China had achieved high quality of economic growth, but in 1992-1996 China had achieved poor quality of economic growth.

Overall, China still has very long road to walk to promote her economy, nonetheless labor productivity growth is crucial for her take-off, particularly in the identified industries such as finance and insurance.

**Conclusion**

This research gives out a new approach to exploring the interindustrial effects of labor productivity for identifying the key sectors to promote overall labor productivity. Since labor is one of the elemental inputs in production and labor productivity is a fundamental figure of economic efficiency, this approach is helpful to make national or regional industrial policy, particularly for the developing countries.

An empirical study on Chinese economy illustrates a comparison between traditional linkage analysis and new approach. The result proves that new approach is needed to study the specific characters of labor productivity. It seems that basic industry such as mining and quarrying, and services such as finance and insurance, have more potential to push labor productivity growth of other sectors and whole economy than manufacturing industry.

Finally, it is demonstrated that Leontief's input-output framework still be quite valuable in theoretical model building and empirical research. Further directions on labor productivity study by presented approach are incorporating Leontief static inverse and modifying it to dynamic model. Of course it will be more practicable if this approach be extended to other productivity issues such as capital, innovation, etc.

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APPENDIX

1. To the column equation

$$X_j = \sum_{i=1}^{n} X_{ij} + D_j + V_J + M_j \tag{1}$$

by using the direct coefficients, we get

$$\sum_{i=1}^{n} a_{ij} + d_j + V_j + m_j = 1 \tag{2}$$

by using the dierct distribution coefficient, we get

$$X_j = \sum_{i=1}^{n} s_{ij} X_i + D_j + V_J + M_j \tag{3}$$

Define

$$\begin{cases} P_j = X_j / L_j \\ \Delta V_j = 0 \quad and \quad \Delta L_j = 0 \end{cases} \tag{4}$$

here $P_j$: labor productivity of sector j

$L_j$: number of staff and workers of sector j

From (3) and (4), the output increase of sector j can be written as

$$\begin{aligned} \Delta X_j &= \sum_{i=1}^{n} \Delta(s_{ij} X_i) + \Delta D_j + \Delta M_j \qquad \text{(suppose } \Delta S_{ij} = 0\text{)}^{(1)} \\ &= \sum_{i=1}^{n} s_{ij} (L_i \Delta P_i) + \Delta D_j + \Delta M_j \end{aligned}$$

Then $$\Delta X = [I - (\hat{D} + \hat{M})]^{-1} S^T \hat{L} \Delta P$$

(5)

here $\hat{D}$ =diag{$d_1$,…, $d_n$}, $\hat{M}$ =diag{$m_1$,…,$m_n$}, $\hat{L}$ =diag{$L_1$,…,$L_n$}

Let $H = [I - (\hat{D} + \hat{M})]^{-1} S^T \quad s_j = 1 - (d_j + m_j) = \sum_{i=1}^{n} a_{ij} + v_j$

then $$H = \begin{pmatrix} \frac{1}{s_1} & & \\ & \ddots & \\ & & \frac{1}{s_n} \end{pmatrix} \begin{pmatrix} s_{11} & \cdots & s_{n1} \\ \vdots & \ddots & \vdots \\ s_{1n} & \cdots & s_{nn} \end{pmatrix} = \begin{pmatrix} \frac{s_{11}}{s_1} & \cdots & \frac{s_{n1}}{s_1} \\ \vdots & \ddots & \vdots \\ \frac{s_{1n}}{s_n} & \cdots & \frac{s_{nn}}{s_n} \end{pmatrix}$$

Define

$$c_j = \frac{s_{1j}}{s_j} + \cdots + \frac{s_{nj}}{s_j} = \frac{\sum_{i=1}^{n} s_{ij}}{v_j (1 + \sum_{i=1}^{n} a_{ij} / v_j)} \tag{6}$$

The numerator of this fraction reflects the level of backward linkage of sector j to other sectors, the denominator reflects the direct cost structure of sector j (only depreciation omitted). Following a simple mathematical deduction the value of denominator will increase as the value of $v_j$ increases. (2)

2. Let $d_j =$ the output increase rate of sector j

$r_j =$ labor productivity increase rate of sector j i.e. $\begin{cases} \Delta X_j = d_j X_j \\ \Delta P_j = r_j P_j = r_j X_j / L_j \end{cases}$

following (4) and (5), we get: [3]

$$\begin{pmatrix} d_1 \\ \dots \\ d_n \end{pmatrix} = \begin{pmatrix} \frac{s_{11}}{s_1} & \dots & \frac{s_{n1}}{s_1} \\ & \dots & \\ \frac{s_{1n}}{s_n} & \dots & \frac{s_{nn}}{s_n} \end{pmatrix} \begin{pmatrix} r_1 \\ \dots \\ r_n \end{pmatrix} = H \begin{pmatrix} r_1 \\ \dots \\ r_n \end{pmatrix} \tag{7}$$

Let $\begin{cases} r_j \neq 0 \\ r_j = 0 \end{cases} (j \neq i; i, j = 1, \dots, n)$

solve (7), we get

$$\begin{cases} d_1 = \frac{s_{i1}}{s_1} r_i \\ \dots \\ d_n = \frac{s_{in}}{s_n} r_i \end{cases} \tag{8}$$

Let $w_j = X_j / \sum_{j=1}^{n} X_j$ (weighted coefficient)

then increase rate of total output

$$d = \sum_{j=1}^{n} w_j d_j = \sum_{j=1}^{n} w_j \frac{s_{ij}}{s_j} r_i = r_i \sum_{j=1}^{n} w_j \frac{s_{ij}}{s_j}$$

therefore $$e_i = \sum_{j=1}^{n} w_j \frac{s_{ij}}{s_j} \tag{9}$$

Note:

(1) $$\Delta(s_{ij} X_i) = \Delta s_{ij} X_i + s_{ij} \Delta X_i + \Delta s_{ij} \Delta X_i \tag{1'}$$

It is always ignored the third term of the right hand in static analysis (second-order infinitesimal). According to the static properties of classical Input-Output model, it is also reasonable to assume that the distribution coefficient keeps constant in short run ($\Delta s_{ij} = 0$), for market generally would not be disturbed abruptly.

(2) Suppose $x \geq 0,\ y \geq 0$

let $z = xy,\ w = |x - y|$,

then $z$ is a decreasing function of $w$.

(3)

$$\begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{pmatrix} \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix} = H\hat{L} \begin{pmatrix} r_1 & & \\ & \ddots & \\ & & r_n \end{pmatrix} \hat{L}^{-1} \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix} = H \begin{pmatrix} r_1 & & \\ & \ddots & \\ & & r_n \end{pmatrix} \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix}$$

i.e. $$\left[ \begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{pmatrix} - H \begin{pmatrix} r_1 & & \\ & \ddots & \\ & & r_n \end{pmatrix} \right] \begin{pmatrix} X_1 \\ \dots \\ X_n \end{pmatrix} = 0 \tag{2'}$$

In accord with the economic sense, $X_j > 0$, the determinant of this homogenous linear equation must be equal to zero:

$$\left| \begin{pmatrix} d_1 & & \\ & \ldots & \\ & & d_n \end{pmatrix} - H \begin{pmatrix} r_1 & & \\ & \ldots & \\ & & r_n \end{pmatrix} \right| = 0 \qquad (3')$$

transforming (3')

$$\begin{vmatrix} d_1 - \frac{s_{11}}{s_1} r_1 & \ldots & -\frac{s_{n1}}{s_1} r_n \\ \vdots & \ddots & \vdots \\ -\frac{s_{1n}}{s_n} r_1 & \ldots & d_n - \frac{s_{nn}}{s_n} r_n \end{vmatrix} = \begin{vmatrix} d_1 - \sum_{i=1}^{n} \frac{s_{i1}}{s_1} r_i & \ldots & -\frac{s_{n1}}{s_1} r_n \\ \vdots & \ddots & \vdots \\ d_n - \sum_{i=1}^{n} \frac{s_{in}}{s_n} r_i & \ldots & d_n - \frac{s_{nn}}{s_n} r_n \end{vmatrix} = 0$$

Without loss of generality, we get

$$\begin{cases} d_1 - \sum_{i=1}^{n} \frac{s_{i1}}{s_1} r_i = 0 \\ \vdots \\ d_n - \sum_{i=1}^{n} \frac{s_{in}}{s_n} r_i = 0 \end{cases}$$

i.e.

$$\begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix} = \begin{pmatrix} \frac{s_{11}}{s_1} & \cdots & \frac{s_{n1}}{s_1} \\ \vdots & \ddots & \vdots \\ \frac{s_{1n}}{s_n} & \ldots & \frac{s_{nn}}{s_n} \end{pmatrix} \begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix} = H \begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix} \qquad (4')$$